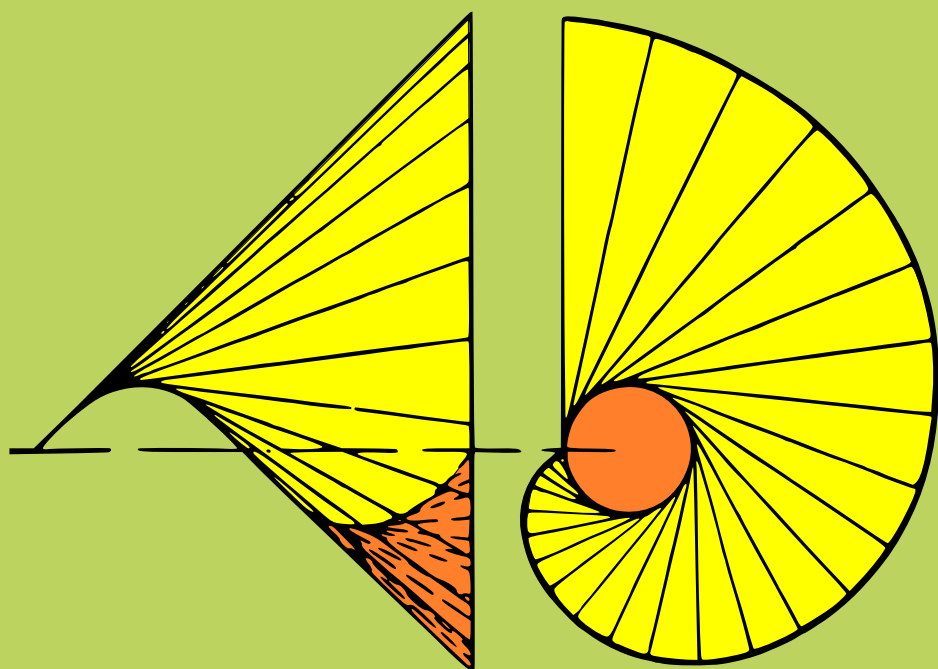


A. T. CHAHLY

# DESCRIPTIVE GEOMETRY



THE HIGHER SCHOOL PUBLISHING HOUSE  
Moscow











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THE HIGHER SCHOOL PUBLISHING HOUSE  
Moscow

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BY A. E. TCHERNUKHIN AND TH. BOTTING

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## AUTHOR'S PREFACE

Descriptive Geometry and Engineering Drawing are among the first engineering subjects taught at institutions of higher education. These subjects are essential if the students are to obtain a fundamental knowledge of the design and construction of machines and mechanisms.

Descriptive geometry, as a specific field of knowledge, like many others, arose and developed as a direct result of man's creative labour.

The remains of fortresses, palaces and other structures found in India, Egypt, Greece, and other countries where there existed ancient culture, show that they were built in accordance with representations resembling modern drawings and conforming to the laws developed by descriptive geometry. A statue discovered during the excavations of ancient Babylon shows a man holding on his knees a slab with a structural drawing on it. The representations of plans and parts of building have also been found on the walls of Egyptian pyramids.

In our times Russian scientists and engineers have played an important role in the development of descriptive geometry and engineering drawing. Descriptive geometry in Russia was taught as early as 1810 in the St. Petersburg Institute of the Road Engineering Corps, now called the Leningrad Institute of Transport Engineering. Russian scientists have written a number of books on descriptive geometry, including such comprehensive courses as *Fundamentals of Descriptive Geometry* by Prof. I. A. Sevostyanov, *Descriptive Geometry* by Prof. P. A. Galaktionov, as well as the textbooks *A Course of Descriptive Geometry, On Curved Lines and Curved Surfaces, Manual of Engineering Drawing* by Prof. A. C. Reder, the books *Complete Course of Descriptive Geometry* and *Course of Linear Perspective Drawing* by Prof. N. N. Makarov. Many other books were written by various authors during the first half of the 19th century.

The Soviet period in the history of methods of representation is marked, above all, by broad research work, better organization and improvements in the methods of teaching descriptive geometry, drawing and technical sketching in technical colleges and institutes.



Soviet scientists working in this field have contributed important theoretical works on diverse aspects of descriptive geometry. One such example is the work carried out on what are known as conditional representations by Prof. N. P. Chetveroukhin, which has led to the establishment of an entirely new branch of descriptive geometry. Others are the work on the problem of the precision of graphical constructions by Prof. D. I. Kargin, on the use of axonometric projections in architectural drawings by Prof. A. I. Viksel, on the use of methods of descriptive geometry in physical and chemical research by Academician N. S. Kournakov, Prof. V. N. Formakovsky and Prof. A. E. Anosov and, in mechanics, the work of the lecturers G. A. Ananov, E. A. Radov and many others.

Scientists who will work in the field of descriptive geometry are being trained at many institutes of the Soviet Union, in Moscow, Leningrad, Kiev, Tbilisi and other cities, where special departments of descriptive geometry and engineering graphics have been organized.

---

## PREFACE

The geometrical properties of objects are usually studied with the help of their representations or drawings. By means of drawings the form of objects may be shown, the dimensions and relative positions of their various geometrical elements may be determined.

Descriptive geometry is devoted to the exposition of the theory and practice required for the solution of problems involving space distances and relationships and the techniques of constructing representations on a plane.

The fundamental objectives of descriptive geometry are:

1. The study of methods of constructing drawings of objects or projects.
2. The study of techniques for determining the forms and dimensions of objects with the help of drawings, i. e., the reading of drawings.
3. The study of the methods of solving problems involving forms in space by means of plane-geometry drawings.

Descriptive geometry is the theoretical foundation of engineering drawing which, in turn, is the preparatory graphical course required for studying all general engineering and special subjects.

Descriptive geometry develops the capacity for imagining objects in space which is essential for reading drawings.

Every engineer must have a good knowledge of the techniques used to represent objects by drawings. He must, so to say, be able to think in three dimensions. Creative engineering without this ability is quite impossible.

In this book descriptive geometry is dealt with step by step. Numerous examples are given to illustrate all theoretical aspects outlined and the methods of constructing projections.

In this work the geometrical elements of objects in space are denoted in the following manner:

Points —  $A, B, C, \dots$

Straight lines —  $a, b, c, \dots$

Segments or portions of straight lines —  $AB, CD, EF, \dots$

Planes —  $\alpha, \beta, \gamma, \delta, \dots$

The principal planes of projection:

the horizontal plane —  $\Pi_1$ ;

the frontal or vertical plane —  $\Pi_2$ ;

the profile plane —  $\Pi_3$ .

The axes of projection or ground lines —  $Ox, Oy, Oz$ .

The traces of oblique planes —  $k, l, m$ .

The trace-projections of projecting planes —  $\alpha_1, \beta_2, \gamma_3, \dots$

New positions of points —  $\bar{A}, \bar{B}, \bar{C}, \dots, \bar{\bar{A}}, \bar{\bar{B}}, \bar{\bar{C}}, \dots$

Sequence of points —  $A', B', \dots, A'', B'', \dots$

Coincidence of two elements —  $a \equiv b, A_1 \equiv B_1, \dots$

This system of notation is further developed:

1. On the horizontal plane of projection  $\Pi_1$  the projections of points —  $A_1, B_1, C_1, D_1, \dots, A'_1, B'_1, A''_1, B''_1, \dots$

The projections of straight lines —  $a_1, b_1, c_1, d_1, \dots$

The projections of portions of lines —  $A_1B_1, C_1D_1, E_1F_1, \dots, A_1A'_1, B_1B'_1, A_1A''_1, \dots$

The horizontal trace of an oblique plane —  $k_1$ .

The horizontal trace-projection of a plane  $\alpha \perp \Pi_1$  —  $\alpha_1$ .

2. On the frontal plane of projection  $\Pi_2$  the projections of points —  $A_2, B_2, C_2, D_2, \dots, A'_2, B'_2, A''_2, B''_2, \dots$

The projections of straight lines —  $a_2, b_2, c_2, d_2, \dots$

The projections of portions of lines —  $A_2B_2, C_2D_2, E_2F_2, \dots, A_2A'_2, B_2B'_2, A_2A''_2, \dots$

The frontal trace of an oblique plane —  $l_2$ .

The frontal trace-projection of a plane  $\alpha \perp \Pi_2$  —  $\alpha_2$ .

3. On the profile plane of projection  $\Pi_3$  the projections of points —  $A_3, B_3, C_3, D_3, \dots, A'_3, B'_3, A''_3, B''_3, \dots$

The projections of straight lines —  $a_3, b_3, c_3, d_3, \dots$

The projections of portions of lines —  $A_3B_3, C_3D_3, E_3F_3, \dots, A_3A'_3, B_3B'_3, A_3A''_3, \dots$

The profile trace of an oblique plane —  $m_3$ .

The profile trace-projection of a plane  $\alpha \perp \Pi_3$  —  $\alpha_3$ .

---

# Chapter I

## TYPES OF PROJECTIONS

### § 1. Central Projections

In descriptive geometry the construction of plane-geometry drawings, representing three dimensional designs of objects, is based on the method of projections\*.

In Fig. 1 points  $A$  and  $S$  are shown situated in space in front of plane  $\Pi^{**}$ . Through these points a straight line is passed to pierce

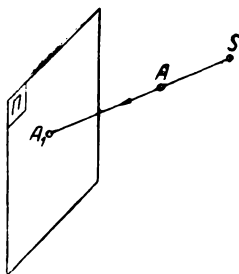


Fig. 1

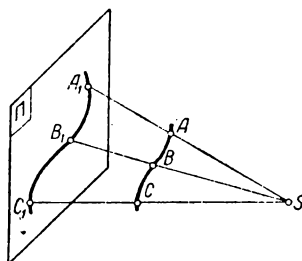


Fig. 2

plane  $\Pi$  in point  $A_1$ . This point is called the *central projection* of point  $A$ . Point  $A$  is termed the *original point*, point  $S$  — the *centre of projection*, the plane  $\Pi$  — the *plane of projection*, and the straight line  $SAA_1$  — the *projecting line* or *projector*.

The centre of projection  $S$  may be also called the *pole* and this type of projection — *polar projection*. A curve  $ABC$  and its central projection  $A_1B_1C_1$  on plane  $\Pi$  are shown in Fig. 2. The bundle of rays originating at the centre  $S$  and projecting the curve  $ABC$  forms a *projecting cone*. Central projection may also be called *conical projection*. The curve  $A_1B_1C_1$  is the intersection of the projecting cone with the plane of projection. It is necessary to understand the concept of central projection, although it is not dealt with in detail in this book.

\* The word "projection" comes from the Latin *projicere* — to hurl or throw.

\*\*  $\Pi$  is the first letter of the Russian word *плоскость* meaning plane.

## § 2. Parallel Projections

If the centre of projections  $S$  is considered to lie at infinity, it follows that all the projecting rays are parallel. In order to draw these rays it is necessary to know the direction of projection. The representations thus obtained are called *parallel projections*.

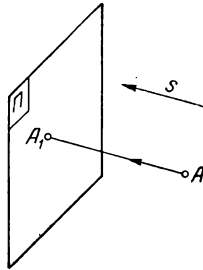


Fig. 3

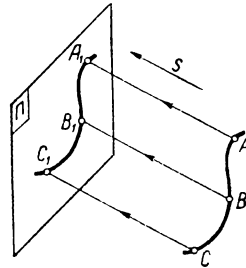


Fig. 4

Fig. 3. shows the construction of the parallel projection  $A_1$  of point  $A$  on the plane of projection  $\Pi$ . The lines to which the projecting ray  $AA_1$  is parallel gives the *direction of projection*.

The parallel rays which project the points of curve  $ABC$  (Fig. 4) form a cylindrical surface, that is why parallel projections are also called *cylindrical projections*.

The given direction of projection may form an oblique angle with the plane of projection or it may be perpendicular to the plane.

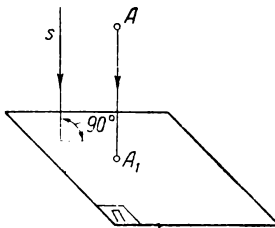


Fig. 5

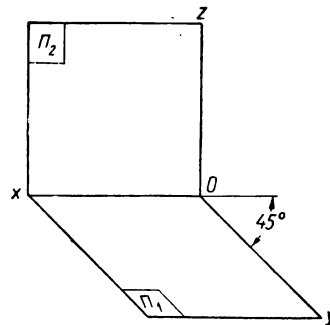


Fig. 6

In the first case the parallel projections are called *oblique projections* (Fig. 4), in the second case *right-angle projections* (Fig. 5).

An example of oblique parallel projection is the representation of a right dihedral or quadrant formed by the *horizontal* plane ( $\Pi_1$ ) and the *vertical frontal* plane ( $\Pi_2$ ) (Fig. 6). This drawing is constructed according to the rules of frontal axonometric projection.\*

\* Axonometric projection, which includes frontal projection, is discussed in Ch. XI.

When representing objects by means of frontal axonometric projection straight lines and their lengths, angles and figures, lying in planes parallel to the frontal plane  $\Pi_2$ , are projected onto that plane  $\Pi_2$  without distortion. As for plane  $\Pi_1$ , the right angles of the quadrant are seen there as angles of  $45^\circ$  and  $135^\circ$ , and the length of

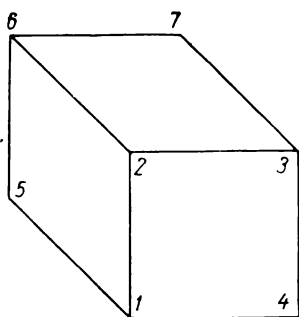


Fig. 7

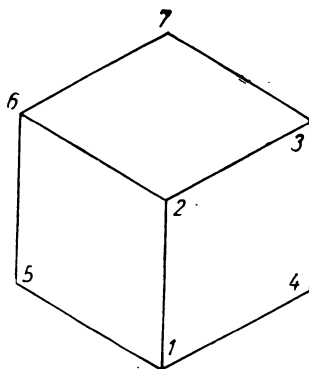


Fig. 8

lines is distorted, unless they are parallel to  $Ox$  and  $Oy$ . This drawing, Fig. 6, will frequently be referred to in this book.

Various ways of representing a cube by means of parallel projection are shown in Figs. 7, 8 and 9. First, this representation is constructed by means of oblique projection onto a plane parallel to the front face of the cube (Fig. 7), then by means of right-angle projection onto a plane inclined to all the sides of the cube (Fig. 8) and, finally, by means of right-angle projection onto a plane parallel to the front face of the cube (Fig. 9).

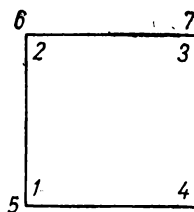


Fig. 9

In engineering practice right-angle projection is most widely used because of its comparative simplicity, the precision of construction it allows, and the possibility of obtaining dimensions without distortion. Right-angle projection is usually termed *orthographic* or *orthogonal projection*.

### § 3. Orthographic Projections

Orthographic projection is illustrated in Fig. 10. The space within the right dihedral or quadrant formed by the horizontal plane  $\Pi_1$  and frontal plane  $\Pi_2$  contains an object having all its faces either parallel or perpendicular to the corresponding plane of projection. For instance, the upper face, and other faces parallel to it, are parallel to plane  $\Pi_1$  and at the same time perpendicular to plane  $\Pi_2$ , and so on.

Let the planes  $\Pi_1$  and  $\Pi_2$  be the planes of projection onto which the object is to be projected. Projecting lines are passed through its vertices perpendicular to the planes of projection. Then the points, where the projectors, perpendicular to plane  $\Pi_1$ , pierce that plane, determine the horizontal projection of the object. The points where the projectors perpendicular to plane  $\Pi_2$ , pierce that plane,

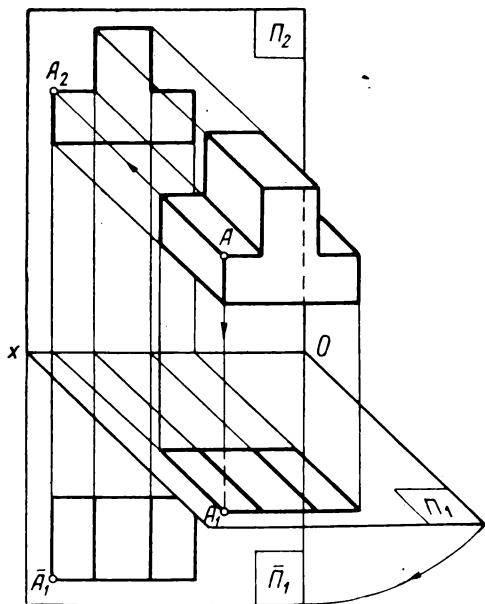


Fig. 10

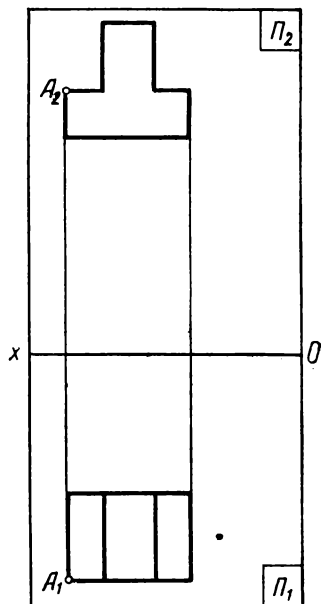


Fig. 11

give the vertical or frontal projection. In Fig. 10 one vertex is indicated by the letter  $A$  and its horizontal and frontal projections by  $A_1$  and  $A_2$ , respectively.

The projections of the object on planes  $\Pi_1$  and  $\Pi_2$  make it possible to visualize the form of the object itself. In order to do this it is necessary to imagine two groups of projectors perpendicular to planes  $\Pi_1$  and  $\Pi_2$ , respectively, and passing through the vertices of the horizontal and frontal projections of the object. Then, the form of the object and its position within the limits of the dihedral, formed by the planes  $\Pi_1$  and  $\Pi_2$ , will become apparent as the result of the intersection of the projectors passing through the two projections of the same points. For instance, point  $A$  is the point of intersection of the projector passing from point  $A_1$  perpendicular to plane  $\Pi_1$  with the projector passing from point  $A_2$  to plane  $\Pi_2$ .

From the drawing in Fig. 10 it can be seen that instead of the right angles, actually formed by the intersection of the edges of the object, acute and obtuse angles are obtained on the horizontal

projection. To avoid such distortions of the right angles on the drawings the plane of projection  $\Pi_1$ , together with the horizontal projection of the object, is rotated about the straight line of intersection of planes  $\Pi_1$  and  $\Pi_2$  — line  $Ox$  — until plane  $\Pi_1$  coincides with plane  $\Pi_2$ . The direction of rotation is indicated by an arrow. This rotation is shown in Fig. 10 as completed. The new position of projection  $A_1$  being  $\bar{A}_1$  and of plane  $\Pi_1$  being  $\bar{\Pi}_1$ . When the horizontal projection of the object is brought into coincidence with plane  $\Pi_2$  there are no distortions of the right angles.

In Fig. 11 the planes  $\Pi_1$  and  $\Pi_2$  are shown as coinciding, and the orthographic projections of the object in these planes of projection are given. Such a representation of an object by two or more of its orthographic projections is called a *multiview drawing*.

---



## Chapter II

### THE POINT

#### § 4. The Multiview Drawing of a Point

The following important facts regarding the projections of a point should be carefully noted:

1. One projection alone does not define the position of a point in space. This can readily be understood from Fig. 12 in which the point  $A$  and its orthographic projection  $A_1$  on plane  $\Pi_1$  are shown. On the projector from point  $A$  per-

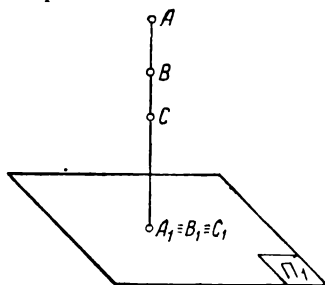


Fig. 12

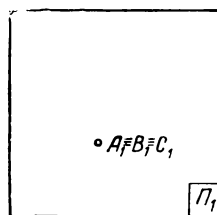


Fig. 13

pendicular to plane  $\Pi_1$  any number of points  $B, C$ , and so on, may be taken, all of which will be projected onto the plane as one and the same point, i. e.,  $A_1 \equiv B_1 \equiv C_1$ . If plane  $\Pi_1$ , with the projection  $A_1 \equiv B_1 \equiv C_1$ , is brought into coincidence with the plane of the drawing (Fig. 13), then the only thing that can be determined is the distance from the base of the perpendicular passing through point  $A$  to the sides of the square that indicates part of the plane of projection  $\Pi_1$ . It is not possible to determine from this drawing (Fig. 13) the altitude of point  $A$  in space in relation to plane  $\Pi_1$ .

2. The position of a point in space may be determined if, instead of a single projection, both projections of a point

on two planes of projection which intersect at right angles are constructed. Two such planes of projection  $\Pi_1$  and  $\Pi_2$  are shown in Fig. 14, a. The line of intersection of the horizontal plane  $\Pi_1$  and the frontal plane  $\Pi_2$  — line  $Ox$  — is called the *ground* or *reference line*. The ground line  $Ox$  divides each of the projection planes into two *flaps*, or *semi-planes*. The horizontal plane  $\Pi_1$  is divided into the

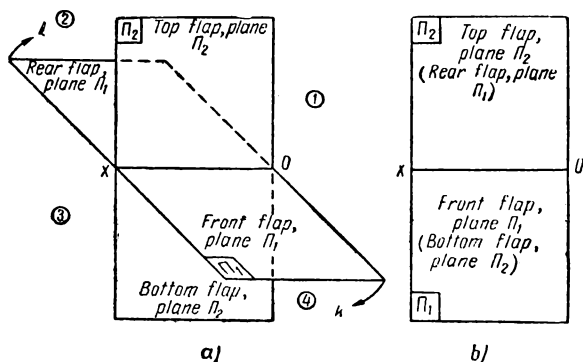


Fig. 14

front and rear flaps and the frontal plane  $\Pi_2$  into the top and bottom flaps.

The planes  $\Pi_1$  and  $\Pi_2$  divide space into four *quarters* or *quadrants*. The first quadrant is limited by the front flap of plane  $\Pi_1$  and the top flap of plane  $\Pi_2$ , the second — by the rear flap of plane  $\Pi_1$  and the top flap of plane  $\Pi_2$ , the third — by the rear flap of plane  $\Pi_1$  and the bottom flap of plane  $\Pi_2$  and, finally, the fourth quadrant is limited by the front flap of plane  $\Pi_1$  and the bottom flap of plane  $\Pi_2$  (Fig. 14, a).

To construct the multiview drawing planes  $\Pi_1$  and  $\Pi_2$  are made to coincide with the plane of the drawing. Assuming plane  $\Pi_2$  to be the plane of the drawing, plane  $\Pi_1$  is rotated about the ground line  $Ox$  so that the front flap of plane  $\Pi_1$  coincides with the lower flap of plane  $\Pi_2$ . The rotation is indicated by arrow  $k$ . The rear flap of plane  $\Pi_1$  is brought into coincidence with the top flap of plane  $\Pi_2$  by rotation in the direction of arrow  $l$  (Fig. 14, b).

3. The orthographic projection of a point onto planes  $\Pi_1$  and  $\Pi_2$  is illustrated in Fig. 15.

The given point  $A$  is shown in the drawing (Fig. 15, a) as lying within the first quadrant. By drawing a perpendicular from point  $A$  onto plane  $\Pi_1$  point  $A_1$  is obtained. This is the horizontal projection of point  $A$ . To obtain the projection of point  $A$  on plane  $\Pi_2$  from point  $A$  a perpendicular is drawn to plane  $\Pi_2$ . The base  $A_2$  of this perpendicular is the frontal projection of point  $A$ .

The plane determined by the projectors  $AA_1$  and  $AA_2$  is

perpendicular both to plane  $\Pi_1$  and plane  $\Pi_2$ . Therefore it is also perpendicular to the ground line  $Ox$  — the line of intersection of  $\Pi_1$  and  $\Pi_2$ . The plane intersects the ground line in a point. Since this point lies in both plane  $\Pi_1$  and plane  $\Pi_2$ , it is denoted by a double subscript as  $A_{12}$ .

The figure  $AA_1A_{12}A_2$  is a rectangle. Therefore, if it is required to find the frontal projection  $A_2$  of point  $A$ , given point  $A$  and its projection  $A_1$ , the procedure is as follows:

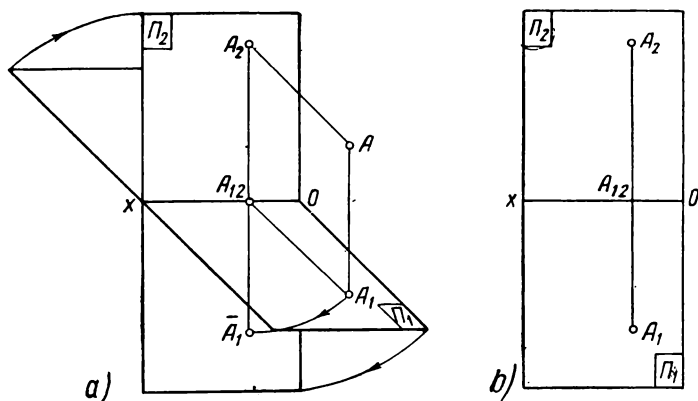


Fig. 15

1. From the horizontal projection  $A_1$  of point  $A$  a perpendicular is drawn to the ground line  $Ox$  which intersects  $Ox$  in point  $A_{12}$ .

2. From  $A_{12}$  in plane  $\Pi_2$  we erect a perpendicular to  $Ox$  to intersect with projector  $AA_2$  in point  $A_2$  which is the required frontal projection of point  $A$ .

To obtain the multiview drawing of point  $A$  plane  $\Pi_1$ , containing projection  $A_1$ , is made to coincide with plane  $\Pi_2$  by rotating it about the line  $Ox$ . The new position of point  $A_1$  (Fig. 15, *a*) is then point  $\bar{A}_1$ . The radius of rotation of point  $A_1$  is the length of line  $A_1A_{12}$  and the centre of rotation is point  $A_{12}$ . Since the rotation of point  $A_1$  takes place in a plane perpendicular to ground line  $Ox$ , then, after rotation, point  $A_1$  will lie on the straight line passing through points  $A_2$  and  $A_{12}$ .

It follows that in the multiview drawing (Fig. 15, *b*) the two projections  $A_1$  and  $A_2$  of the original point  $A$  are located on one and the same line  $A_1A_2$  perpendicular to the ground line  $Ox$ . This line is called a *vertical projector*.

Let us take note of the fact that the portions of the straight lines  $A_1A_{12}$  and  $AA_2$  are equal, as are  $A_2A_{12}$  and  $AA_1$ , since they are opposite sides of a rectangle (Fig. 15, *a*). The length of  $AA_2$  determines the distance from point  $A$  to the frontal plane  $\Pi_2$ . On the multiview drawing (Fig. 15, *b*) the distance of point  $A$

from plane  $\Pi_2$  is given by the length of  $A_1A_{12}$ , i. e., the distance from the horizontal projection  $A_1$  of the point to the ground line  $Ox$ .

The distance of point  $A$  from the horizontal plane of projection  $\Pi_1$  — the altitude of point  $A$  — is equal to  $A_2A_{12}$  (Fig. 15, *a*). On the multiview drawing (Fig. 15, *b*) the distance of point  $A$  from plane  $\Pi_1$  is given by the length of line  $A_2A_{12}$ , i. e., by the distance from the frontal projection  $A_2$  of the point to ground line  $Ox$ .

4. In order to read the multiview drawing of point  $A$  (Fig. 15, *b*), i. e., to establish the position of the point in space in relation to the planes of projection  $\Pi_1$  and  $\Pi_2$ , one should imagine that the drawing is folded along the ground line  $Ox$  until the planes  $\Pi_1$  and  $\Pi_2$  are perpendicular. The lower part of the drawing then occupies the position of the front flap of the plane  $\Pi_1$ , and the top part of the drawing — that of the top flap of the plane  $\Pi_2$ . Then imagine that through points  $A_1$  and  $A_2$  lines, perpendicular to the respective planes  $\Pi_1$  and  $\Pi_2$ , are drawn. The intersection of these perpendiculars determines the position of point  $A$  in space.

## § 5. The Projection of Different Points in Relation to the Planes of Projection

### 1. Points situated in the 2, 3 and 4 quadrants.

A point in a system of two planes of projection  $\Pi_1$  and  $\Pi_2$  may lie in one of the four space quadrants, on one of the plane flaps or, finally, on ground line  $Ox$ .

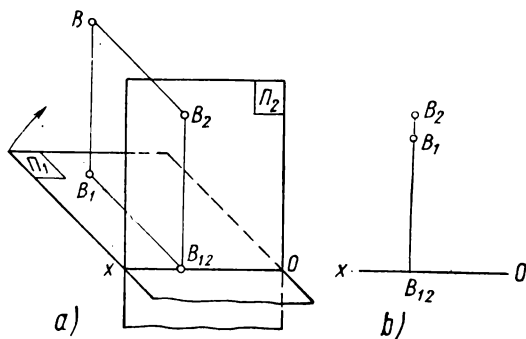


Fig. 16

We have already seen one of the possible positions of a point. Point  $A$  was located in the first quadrant (Fig. 15, *a, b*). The distinguishing feature of the multiview drawing of such a point (Fig. 15, *b*) is that its horizontal projection lies under the ground line  $Ox$ , while the frontal projection lies above this ground line.

In the case of point  $B$ , which is in the second quadrant

(Fig. 16, *a*), both projections on the multiview drawing are located above the ground line  $Ox$  (Fig. 16, *b*), because the horizontal projection  $B_1$  of this point moves with the rear flap of plane  $\Pi_1$  which is made to coincide with the top flap of plane  $\Pi_2$ .

Point  $C$  is located in the third space quadrant (Fig. 17, *a*). On the multiview drawing (Fig. 17, *b*) the horizontal projection  $C_1$  of

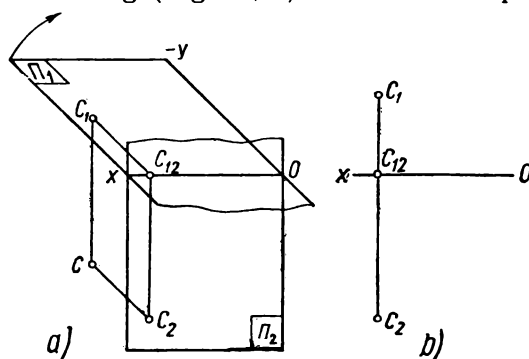


Fig. 17

point  $C$  is located above line  $Ox$ , since after rotation  $C_1$  will lie in the top flap of plane  $\Pi_2$ , while the frontal projection  $C_2$  lies under ground line  $Ox$ .

Point  $D$  (Fig. 18, *a*), which is in the fourth space quadrant, has both its projections on the multiview drawing (Fig. 18, *b*) located under the ground line  $Ox$ , since the horizontal projection  $D_1$  of point  $D$  coincides with the lower flap of plane  $\Pi_2$ .

On the combined multiview drawing (Fig. 19) the projections of all the points  $A$ ,  $B$ ,  $C$  and  $D$  are shown for the purpose of comparison.

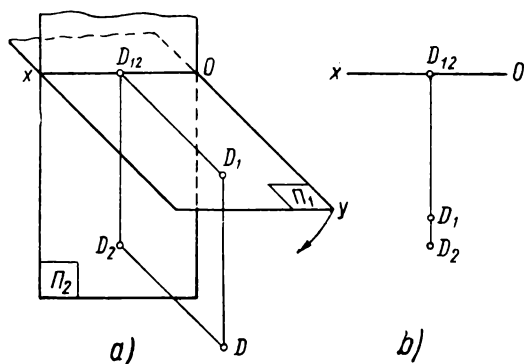


Fig. 18

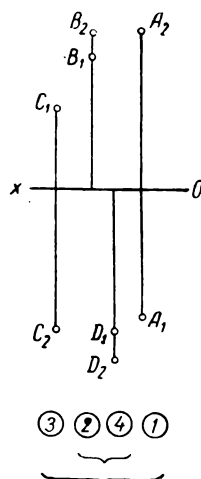


Fig. 19

The horizontal and frontal projections of points  $A$  and  $C$  in the odd-number space quadrants (1 and 3) are located on opposite sides of ground line  $Ox$ , where as both projections of each of the points  $B$  and  $D$ , in the even-number space quadrants (2 and 4), lie on one and the same side of the ground line  $Ox$ .

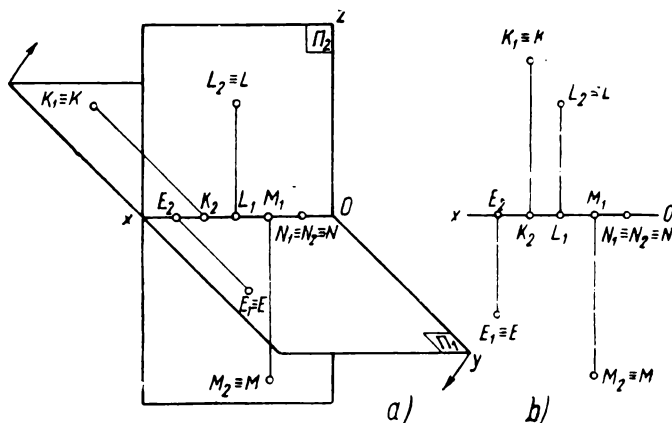


Fig 20

2. The next group of points consists of points, lying in the planes of projection  $\Pi_1$  and  $\Pi_2$ . When their position in space is being determined it is necessary to take into account that the distance from these points to the planes on which they lie is zero. Therefore one of the two projections of these points on the multiview drawing must lie on ground line  $Ox$ , whereas the second projection will coincide with the original point itself.

For example, if point  $E$  (Fig. 20, *a*) lies on the front flap of plane  $\Pi_1$ , then its frontal projection  $E_2$  in the multiview drawing (Fig. 20, *b*) is located on the ground line  $Ox$  and the horizontal projection  $E_1$  coincides with point  $E$  ( $E_1 \equiv E$ ).

In Fig. 20, *a* point  $K$  lies in the rear flap of plane  $\Pi_1$ , point  $L$  --- in the top flap of plane  $\Pi_2$ , and point  $M$  in the lower flap of plane  $\Pi_2$ . One of the projections of each of the points  $K$ ,  $L$  and  $M$  coincides with the point itself, whereas the second, lies on ground line  $Ox$ .

3. Finally, when point  $N$  is located on ground line  $Ox$ , both its projections  $N_1$  and  $N_2$  coincide with the original point  $N$ .

In the multiview drawing Fig. 20, *b* are shown the projections of points  $E$ ,  $K$ ,  $L$ ,  $M$  and  $N$ .

From what has been outlined above the following conclusions may be drawn:

1. If the horizontal projection of a point in the multiview drawing lies below the ground line  $Ox$ , then the point itself is located in space in front of the frontal plane of projection  $\Pi_2$ . If the projection lies above the ground line, then the point itself lies behind plane  $\Pi_2$ .

2. If the frontal projection of a point lies above the ground line  $Ox$ , then the point itself is situated above the horizontal plane of projection  $\Pi_1$ . If this projection lies below the ground line, the point itself lies under this plane.

3. If one of the projections of the point lies on the ground line  $Ox$ , then the original point lies in one of the planes of projection  $\Pi_1$  or  $\Pi_2$ .

4. If both projections of the point coincide and lie on ground line  $Ox$ , then the original point also lies on this line.

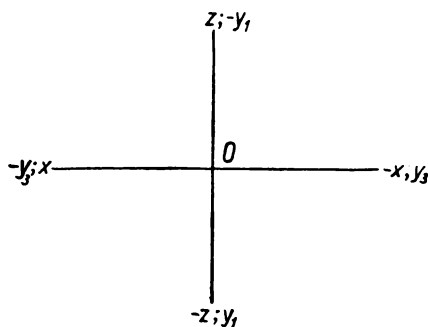
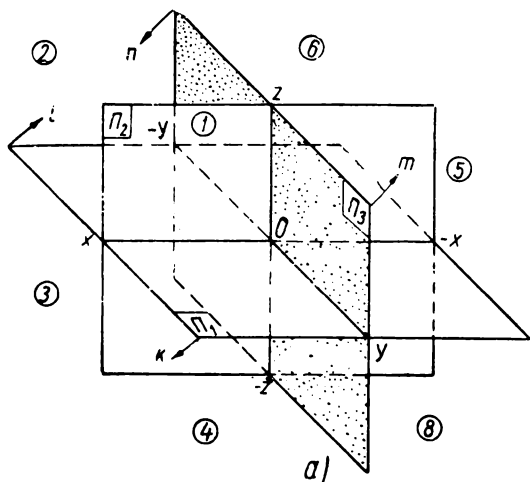


Fig 21

4. Projections on three planes. Let us now examine the orthographic projection of points on three mutually perpendicular planes of projection, introducing the third plane  $\Pi_3$  (Fig. 21, a).

The plane  $\Pi_3$  is called the *profile* plane of projection and the projections on it are called *profile* projections. The profile

plane of projection  $\Pi_3$  intersects the planes of projection  $\Pi_1$  and  $\Pi_2$ , respectively, in the ground lines  $Oy$  and  $Oz$ , both of which are perpendicular to ground line  $Ox$ . From the drawing (Fig. 21, a) it may be seen, that  $Ox \perp \Pi_3$ ,  $Oy \perp \Pi_2$  and  $Oz \perp \Pi_1$ .

The planes of projection  $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$  divide space into eight parts called *octants*. There are two systems of numerating the octants —

right-hand and left-hand. In this text-book the right-hand system is used throughout, since the disposition of the views or projections then corresponds to that accepted in engineering drawing courses. The sequence of numeration using the right-hand system of octants is shown in Fig. 21, *a*. The sequence of the first four octants (1, 2, 3 and 4) coincides with the sequence of the four quadrants formed by the intersection of the planes of projection  $\Pi_1$  and  $\Pi_2$ . The octants 5, 6, 7 and 8 are situated to the right of the plane  $\Pi_3$  in the same order as the octants 1, 2, 3 and 4. In Fig. 21, *a* all the octants are numbered except the seventh, situated under plane  $\Pi_1$  behind plane  $\Pi_2$  and to the right of plane  $\Pi_3$ .

Each of the ground lines is divided by the planes  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  into two parts. Generally the letter *O* is used to indicate the point of intersection of the ground lines. The direction of the line *x* to the left of point *O* is assumed to be positive, and to the right of *O* — negative. The direction of the portion of the ground line *Oy* which is towards the observer is taken as positive and that of the opposite portion away from the observer, as negative. Similarly the direction of the upper part of *Oz*, i. e., above point *O*, is taken as positive and that of the lower part, as negative.

The transition from space representation of the three planes of projection to the multiview drawing is carried out as follows. First the horizontal projection plane  $\Pi_1$  is revolved about the ground line *Ox* until it coincides with the frontal plane of projection  $\Pi_2$ . This rotation is carried out in a clockwise direction, viewing *x* from the left, as shown in Fig. 21, *a* by arrows *k* and *l*. Then the profile plane of projection  $\Pi_3$  is rotated about ground line *Oz* until it coincides with the frontal plane  $\Pi_2$ . This second rotation is carried out counter-clockwise, the axis *z* viewed from the top (shown by arrows *m* and *n*). Thus we obtain the multiview drawing of the planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  (Fig. 21, *b*). Since the frontal plane of projection  $\Pi_2$  is taken as the plane of the drawing and remains stationary, the ground lines *Ox* and *Oz* contained in it will not change their positions on the multiview drawing.

When the plane  $\Pi_1$  coincides with the plane  $\Pi_2$ , the positive direction of the *Oy* line coincides with the negative direction of the *Oz* line, and conversely, the negative direction of the *Oy* line coincides with the positive direction of the *Oz* line.

5. It is often convenient to establish the position of points by giving their coordinates relative to the system of three coordinate planes. If the point *A* is assigned by rectangular coordinates, the planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  serve as the planes of the system of coordinates. The point *O* of intersection of axes *x*, *y* and *z* is then the point of origin (Fig. 22, *a*). The coordinates of point *A* are three values *x*, *y* and *z*, when: 1) the absolute magnitudes of *x*, *y* and *z* are respec-



tively equal to the lengths  $AA_3$ ,  $AA_2$ ,  $AA_1$ , i. e., the distances of point  $A$  from the planes  $\Pi_3$ ,  $\Pi_2$  and  $\Pi_1$ ; 2)  $x > 0$ , if  $A$  lies to the left of  $\Pi_3$ ;  $x < 0$ , if  $A$  lies to the right of  $\Pi_3$ ; 3)  $y > 0$ , if  $A$  lies in front of  $\Pi_2$ ;  $y < 0$ , if  $A$  lies behind  $\Pi_2$ ; 4)  $z > 0$ , if  $A$  lies above  $\Pi_1$ ;  $z < 0$ , if  $A$  lies under  $\Pi_1$ . Knowing the coordinates of a point it is possible to construct the projection of the point and, conversely, by means

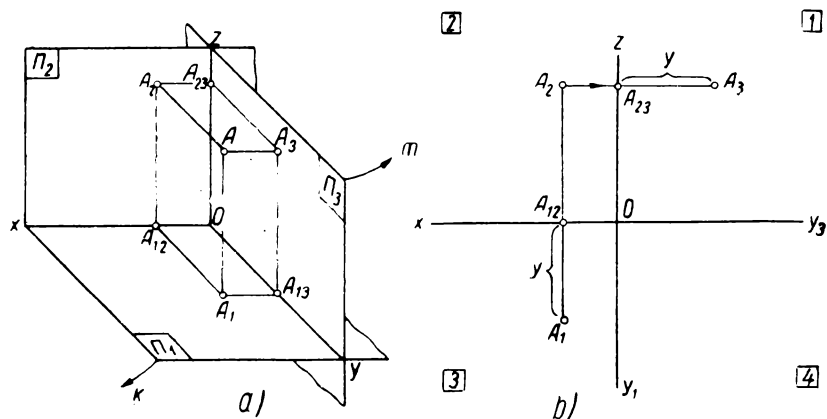


Fig. 22

of the multiview drawing the coordinates of a point may be determined.

The planes  $AA_1A_2$ ,  $AA_2A_3$ ,  $AA_3A_1$ ,  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  form the rectangular *coordinate parallelepiped*.

The coordinate signs for all the octants are given in the table.

| Octants | Coordinates              |     |     |
|---------|--------------------------|-----|-----|
|         | $x$                      | $y$ | $z$ |
|         | Signs of the Coordinates |     |     |
| 1       | +                        | +   | +   |
| 2       | +                        | -   | +   |
| 3       | +                        | -   | -   |
| 4       | -                        | +   | -   |
| 5       | -                        | +   | +   |
| 6       | -                        | -   | +   |
| 7       | -                        | -   | -   |
| 8       | -                        | +   | -   |

6. In Fig. 22, a point  $A$  is situated in the first octant. Its coordinates  $x$ ,  $y$ ,  $z$  are respectively equal to 7, 13 and 18 mm. This should be checked from the drawing. The construction of the coordinate parallelepiped of this point consists in marking the points  $A_{12}$ ,  $A_{13}$  and  $A_{23}$  on the lines  $Ox$ ,  $Oy$  and  $Oz$ . This is done by stepping off a distance of 7 mm from point  $O$  on the ground line  $Ox$ , a distance of 13 mm on the ground line  $Oy$

and a distance of 18 mm on  $Oz$ . The other edges of the parallelepiped are drawn parallel to these segments.

It should be noted that any two projections of point  $A$  are con-

tained in each of the faces of the coordinate parallelepiped, when this face does not coincide with the plane of projection. For example, the horizontal and frontal projections,  $A_1$  and  $A_2$ , lie in the profile face  $AA_1A_{12}A_2$ . The horizontal and profile projections,  $A_1$  and  $A_3$ , lie in the frontal face  $AA_1A_{13}A_3$  and, finally, the frontal and profile projections,  $A_2$  and  $A_3$ , lie in the horizontal face  $AA_2A_{23}A_3$  of the parallelepiped. Therefore, to obtain two projections of a point, given its coordinates  $x$ ,  $y$  and  $z$ , it is necessary to construct only the corresponding face of the coordinate parallelepiped. The projections  $A_1$  and  $A_2$  of point  $A$  (7; 13; 18) are shown in Fig. 23. First from point  $O$  on the line  $Ox$  mark off the distance  $x=7$  mm ( $OA_{12}=7$  mm). After that on a line, drawn from  $A_{12}$  parallel to the line  $Oy$ , the distance  $A_{12}A_1=13$  mm is marked off. Finally, from point  $A_{12}$  a line is drawn parallel to axis  $Oz$ , and the distance  $A_{12}A_2=18$  mm is marked off on it.

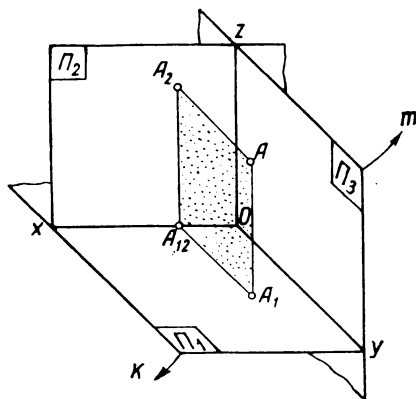


Fig. 23

If it is necessary to construct a point by its assigned coordinates, this may be done by constructing one *coordinate open polygon*. Thus in Fig. 23 point  $A$  and its projection  $A_1$  are obtained by drawing the coordinate open polygon  $OA_{12}A_1A$ .

The construction of the horizontal and frontal projections of point  $A$  in the multiview drawing (Fig. 22, *b*) is the same as for any point located in the first space quadrant (Fig. 15, *b*). The construction of the profile projection  $A_3$  of point  $A$  is carried out as follows: 1) through point  $A_2$  draw a horizontal projector, which is the line of altitude, and 2) from point  $A_{23}$  lay off along this line, to the right of  $Oz$ , distance  $A_{23}A_3=13$  mm ( $y=13$ ).

Point  $A_3$  is the profile projection of point  $A$ . This is so because 1) the distance of the profile projection  $A_3$  from the ground line  $Oy_3$ , which coincides with the  $x$  axis, is the same as the distance from frontal projection  $A_2$  to the  $x$  axis, and 2)  $A_{23}A_3$  and  $A_1A_{12}$  are equal to  $y$ , given by the coordinates of point  $A$ .

7. Let us now consider the construction of the multiview drawings of points located in the other seven octants. Point  $B$  lies in the second octant (Fig. 24). Its projections  $B_1B_2$  and  $B_3$  in Fig. 24, *a* are obtained by constructing the profile face ( $BB_1B_{12}B_2$ ) and the horizontal face ( $BB_2B_{23}B_3$ ) of the coordinate parallelepiped.

In the multiview drawing (Fig. 24, *b*) the two principal projec-

tions  $B_1$  and  $B_2$  of point  $B$  are located to the left of point  $O$  and above ground line  $Ox$ . The construction of these projections corresponds to that of points  $B_1$  and  $B_2$  in Fig. 16,  $b$ .

The profile projection  $B_3$  is located to the left of  $Oz$  since, after the plane  $\Pi_3$  has been revolved about the ground line  $Oz$ , the rear half of this plane coincides with the left half of plane  $\Pi_2$ . The

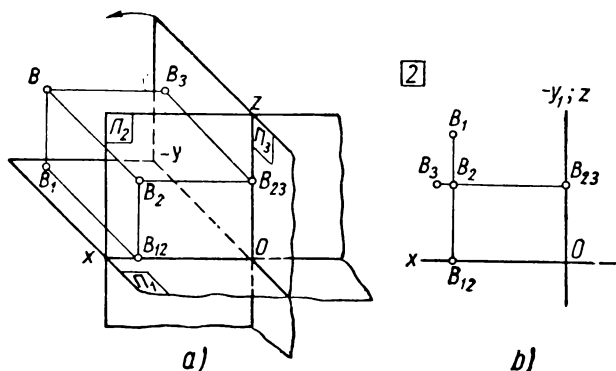


Fig. 24

construction of the profile projection  $B_3$  of point  $B$  is carried out as follows: 1) through the frontal projection  $B_2$  a horizontal projector is drawn parallel to  $Ox$ ; 2) on this projector, to the left of  $Oz$ , lay off the distance  $y$  equal to the distance  $(B_1B_{12})$  at which projection  $B_1$  lies from the ground line  $Ox$ .

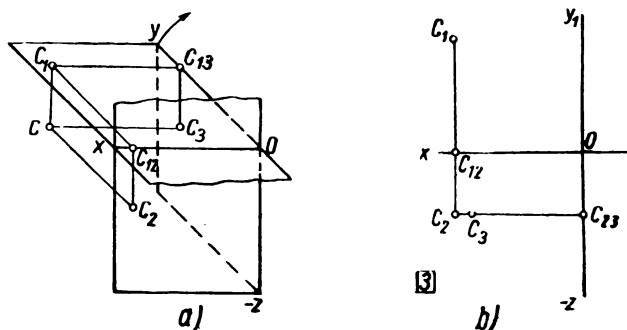


Fig. 25

Point  $C$  lies in the third octant (Fig. 25,  $a$ ) and in the multi-view drawing (Fig. 25,  $b$ ) the positions of its principal projections  $C_1$  and  $C_2$  are the same as those of the projections of a point in the third quadrant (Fig. 17,  $b$ ), namely, the horizontal projection lies above the ground line  $Ox$ , and the frontal projection lies under  $Ox$ .

To construct the profile projection  $C_3$  by means of these two projections the horizontal projector of point  $C$  is drawn through point  $C_2$ . On it, to the left of  $Oz$ , mark off the negative distance  $-y$ , which is equal to the length of  $C_1C_{12}$ .

Point  $D$  is situated in the fourth octant (Fig. 26). Its projections  $D_1$ ,  $D_2$  and  $D_3$  in the drawing (Fig. 26, *a*) are constructed with

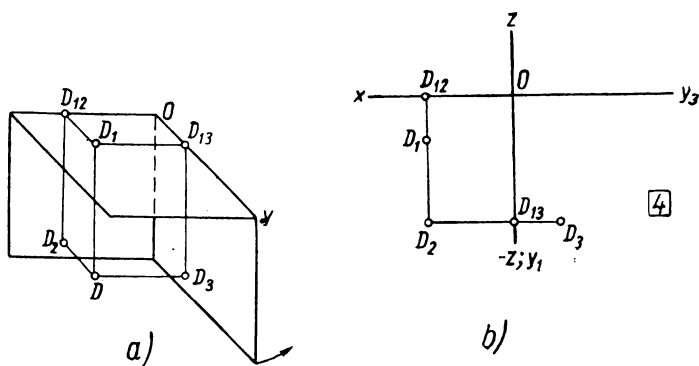


Fig. 26

the help of the profile face ( $DD_1D_{12}D_2$ ) of the coordinate parallelepiped and the frontal face ( $DD_1D_{13}D_3$ ).

The principal projections  $D_1$  and  $D_2$  of point  $D$  in the multiview drawing (Fig. 26, *b*) are constructed in full accordance with the construction of a point situated in the fourth quadrant (Fig. 18, *b*). To construct the profile projections  $D_3$  of point  $D$  by means of these projections, through the frontal projection  $D_2$  a horizontal projector is drawn and on it, to the right of  $Oz$ , the positive coordinate  $y$ , which is equal to  $D_1D_{12}$ , is laid off. From the multiview drawing of points  $A$ ,  $B$ ,  $C$  and  $D$  situated in the 1, 2, 3 and 4 octants, respectively, it can easily be seen that the profile projections of these points are located in the parts of the multiview drawing (divided by the  $x$ ,  $y$ ,  $z$  axes, as shown in Fig. 22, *b*) the numbers of which 1, 2, 3 and 4 coincide with the numbers of the octants. So projection  $A_3$  of point  $A$  of the first octant lies in the top right part 1 of the multiview drawing (Fig. 22, *b*). The projection  $B_3$  of point  $B$  of the second octant lies in the top left part 2 of the drawing (Fig. 24, *b*). The projection  $C_3$  of point  $C$  of the third octant lies in the lower left part 3 of the drawing (Fig. 25, *b*). Finally, the projection  $D_3$  of point  $D$  of the fourth octant lies in the lower right part 4 of the drawing (Fig. 26, *b*). This will help the student to orientate himself more quickly when constructing the three projections of points situated in the first four octants. It will also

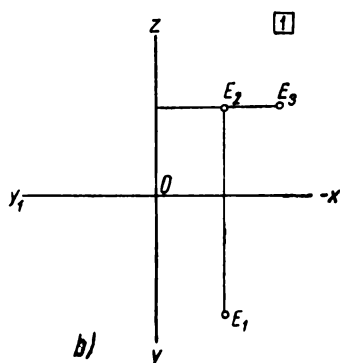
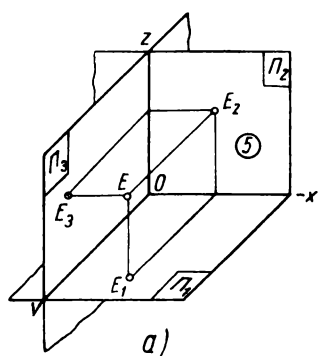


Fig. 27

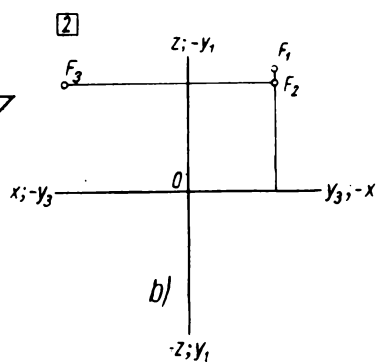
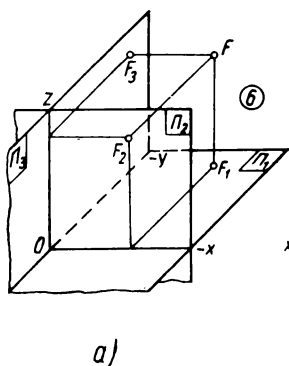


Fig. 28

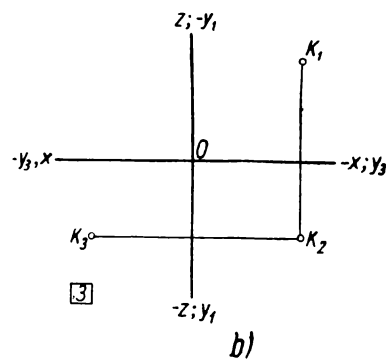
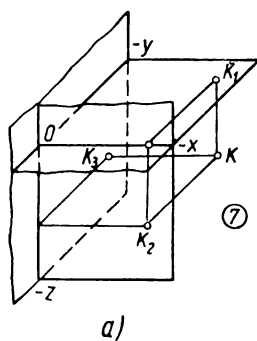


Fig. 29

help him to determine, by means of the multiview drawing, in which octant a point is located in space.

The points  $E$ ,  $F$ ,  $K$  and  $L$ , situated respectively in the fifth, sixth, seventh and eighth octants, are shown in Figs. 27 to 30. The sequence of the construction of their multiview drawings is the same as for points  $A$ ,  $B$ ,  $C$  and  $D$ , i. e., the principal projections of

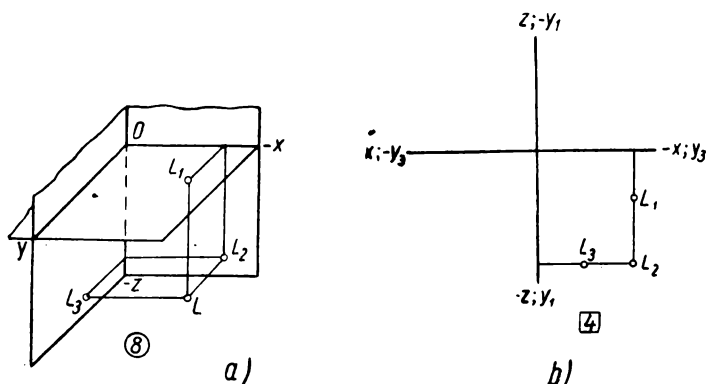


Fig. 30

these points on the planes  $\Pi_1$  and  $\Pi_2$  are constructed first, and then their projections on plane  $\Pi_3$  are found using the method already outlined.

It should be noted that:

1. The horizontal and frontal projections of points which are located in the 5, 6, 7 and 8 octants are seen in the multiview drawing in the same order as the points of the 1, 2, 3 and 4 octants, correspondingly located in space. In other words, the projections of points in the odd-numbered octants (5 and 7) are located on both sides of the ground line  $Ox$ , the projections of points in the even-numbered octants (6 and 8) are both on one side of  $Ox$ .

2. The profile projections of points located in the 5, 6, 7 and 8 octants are situated in those parts of the multiview drawing (divided by lines  $x$ ,  $y$ ,  $z$ , as shown in Fig. 22, b), the numbers of which are obtained by subtracting 4 from the number of the corresponding octant. In this way the projection  $E_3$  of point  $E$  in the fifth octant (Fig. 27, b) is situated in the top right-hand part  $\boxed{1}$  of the multiview drawing. The projection  $F_3$  of point  $F$  in the sixth octant (Fig. 28, b) lies in the top left part  $\boxed{2}$  of the drawing. The projection  $K_3$  of point  $K$  in the seventh octant (Fig. 29, b) lies in the lower left side  $\boxed{3}$  of the drawing and, finally, the projection  $L_3$  of point  $L$  in the eighth octant (Fig. 30, b) is situated in the lower right part  $\boxed{4}$  of the drawing.

3. Since the values of  $x$  for points located in the 5, 6, 7 and 8 octants are negative, the horizontal and frontal projections of these points in the multiview drawing are situated to the right of  $Ox$ .

### Problems and Exercises

1. Does one projection of a point determine the position of that point in space?

2. Prove that two projections of a point completely determine its position in space.

3. Prove that in the multiview drawing two projections of a point are located on one line perpendicular to the ground line.

4. How is the distance from plane  $\Pi_1$  of the projection of a point determined in the multiview drawing?

5. How is the distance from plane  $\Pi_2$  of the projection of a point determined in the multiview drawing?

6. What is the position in space, in relation to plane  $\Pi_1$ , of a point, the frontal projection of which is a) above the ground line  $Ox$ , and b) below it?

7. What is the position in space relative to plane  $\Pi_2$  of a point, the horizontal projection of which is a) below the ground line  $Ox$ , and b) above it?

8. How are the principal projections situated in the multiview drawing of a point which is a) in the first quadrant, b) in the second quadrant, c) in the third quadrant, and d) in the fourth quadrant?

9. How are the principal projections located in relation to the ground line  $Ox$  in the multiview drawing of a point which lies a) in the front flap of plane  $\Pi_1$ , b) in the top flap of plane  $\Pi_2$ , c) in the bottom flap of plane  $\Pi_2$ , and d) on the ground line  $Ox$ ?

10. How is the profile projection of a point determined in the multiview drawing by means of its two principal projections on planes  $\Pi_1$  and  $\Pi_2$ ?

11. In which octant is a point located if a) its coordinates  $x$ ,  $y$  and  $z$  are positive, if b)  $x$  and  $z$  are positive and  $y$  negative, and if c)  $x$  is negative and  $y$  and  $z$  positive?

12. Which are the positive and which are the negative coordinates of points located in the a) second, b) fifth, c) seventh, d) eighth octant?

13. Construct the three-dimensional drawing and the multiview drawing of points  $A$  (5; 15; 7),  $B$  (10; — 15; 20),  $C$  (15; — 10; — 12) and  $D$  (20; 8; — 16). The coordinates are given in mm.

Hint. First determine in which octant the given points are located, imagining the broken coordinate line for each of the points (see Fig. 23 and the corresponding explanation).

14. Write down the coordinates of the points, given in Fig. 31, and state in which space quadrants these points are located.

Hint. It is necessary to imagine that the drawing is folded along ground line  $Ox$  as in Fig. 15, *b* and that through the projections of each point perpendiculars are drawn to the planes of projections  $\Pi_1$  and  $\Pi_2$  respectively.

15. Given point  $A$  (20; 10; 15), construct point  $B$  symmetrical to the given point  $A$  in relation to plane  $\Pi_2$ , point  $C$  symmetrical to point  $A$  in relation to plane  $\Pi_1$ , and point  $D$  symmetrical to point  $A$  in relation to the ground line  $Ox$ .

Hint. Point  $A$  is located in the first quadrant (all the three coordinates  $x$ ,  $y$  and  $z$  are positive). Therefore point  $B$  lies in the second quadrant, point  $C$  in the fourth and point  $D$  in the third.

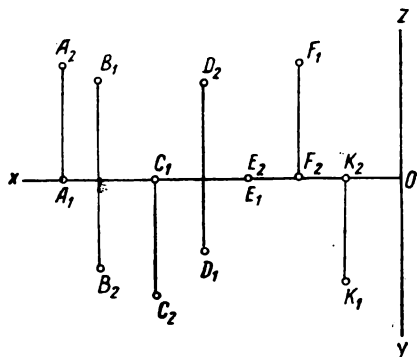


Fig. 31

16. Construct the three-dimensional view and the multi-view drawing of points  $A$ ,  $B$  and  $C$ , point  $A$  being in the second octant, point  $B$  in the fifth and point  $C$  in the seventh.

17. Write down the coordinates of the points  $A$ ,  $B$  and  $C$  and state in which octants these points are located (Fig. 32).

Hint. The text explaining Figs. 22, 24, 25 and 26 should be studied.

18. By means of the pairs of projections of points  $A$  and  $B$ , construct their third projection and indicate the octants in which these points lie (Fig. 33).

Hint. It should be borne in mind that in the multiview drawing the horizontal and frontal projections of any point lie on the vertical

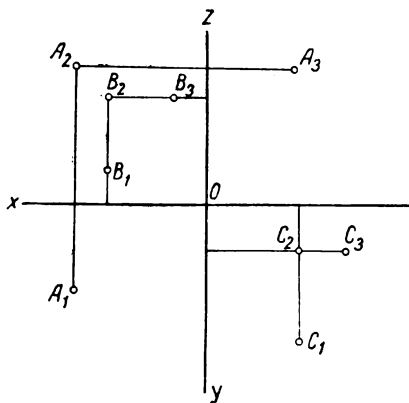


Fig. 32

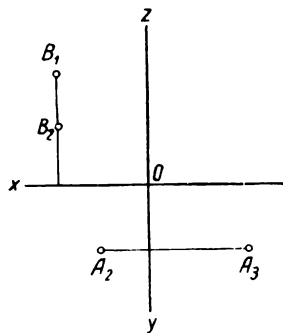


Fig. 33



projector, i. e., on the line perpendicular to ground line  $Ox$ , and the frontal and profile projections on the horizontal line of altitude on a line perpendicular to  $Oz$ .

It should also be recalled that  $+y$  on the vertical projector is laid off from line  $Ox$  below that line and  $-y$  above it. On the horizontal projector that same distance  $+y$  is laid off to the right of line  $Oz$ , and  $-y$  to the left of that line.

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### Chapter III

## THE STRAIGHT LINE

### § 6. The Multiview Drawing of a Straight Line

1. In order to obtain the orthographic projection of a straight line on a plane, through the line a plane perpendicular to the plane of projection is passed. The line of intersection of this plane with the plane of projection is then the orthographic projection of the line.

In Fig. 34 through a given line  $AB$  a plane  $\alpha$  perpendicular to plane  $\Pi_1$  is passed. The intersection  $A_1B_1$  of plane  $\alpha$  with plane  $\Pi_1$  determines the only possible projection of line  $AB^*$  on plane  $\Pi_1$ .

Since a line is completely determined by any two of its points, instead of drawing a plane, it is sufficient to find the orthographic projections  $A_1$  and  $B_1$  of points  $A$  and  $B$  on plane  $\Pi_1$ . The line joining points  $A_1$  and  $B_1$  is the orthographic projection of  $AB$  on plane  $\Pi_1$ .

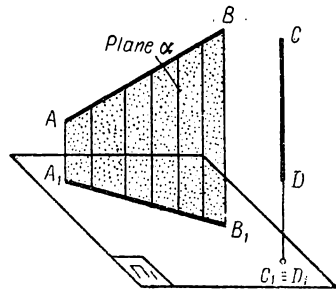


Fig. 34

In Fig. 34 a line  $CD$  perpendicular to plane  $\Pi_1$  is also shown. Perpendiculars to plane  $\Pi_1$ , passing through points  $C$  and  $D$ , will coincide with the line itself and intersect plane  $\Pi_1$  in one point  $C_1 \equiv D_1$ .

It follows that the orthographic projection of a straight line on a plane is a straight line except when this line is perpendicular to the plane of projection.

Similarly any straight line in a plane may be considered as being the projection of a straight line.

\* The word line in this book may be assumed to imply a straight line, unless otherwise stated.

Let a plane  $\alpha$  perpendicular to  $\Pi_1$  (Fig. 34) be passed through the straight line  $A_1B_1$  which lies in plane  $\Pi_1$ . Then  $AB$  will be the projection on plane  $\Pi_1$  of a line, not perpendicular to  $\Pi_1$ , for example  $AB$ , drawn in plane  $\alpha$ .

2. A single projection of a line does not determine the position of that line in space, since it is

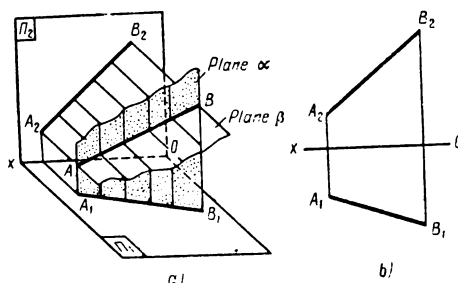


Fig. 35

possible to draw in the plane  $\alpha$  (Fig. 34) any number of lines both parallel to  $\Pi_1$  or inclined to that plane, while all their projections on plane  $\Pi_1$  coincide with one line, i. e.,  $A_1B_1$ .

3. Two projections of a line fully determine its position in space.

Knowing the position of two projections of a line — the horizontal  $A_1B_1$  and the frontal  $A_2B_2$  (Fig. 35, a) it is possible to pass through  $A_1B_1$  a plane  $\alpha$  perpendicular to plane  $\Pi_1$  and through  $A_2B_2$  a plane  $\beta$  perpendicular to plane  $\Pi_2$ . The line of intersection  $AB$  of the planes  $\alpha$  and  $\beta$  determines the one and only position in space of the line, the projections of which were given.

Since two points fully determine the position of a line in space, the construction of the multiview drawing of a line is therefore reduced to the construction of the projection of two of its points. It is common practice to assume the extreme points as these points, if the line represented is of a given length.

To construct the multiview drawing of the line  $AB$  (Fig. 35, a) the coordinates  $x$ ,  $y$  and  $z$  of points  $A$  and  $B$  are determined and transferred to Fig. 35, b. The pairs of projections obtained, i. e.,  $A_1$  and  $B_1$  and  $A_2$  and  $B_2$  are then joined up. Lines  $A_1B_1$  and  $A_2B_2$  are, respectively, the horizontal and frontal projections of line  $AB$ .

Example. It is required to construct the multiview and the three-dimensional drawings of line  $AB$  by means of the coordinates of its points  $A$  (20; 15; 7) and  $B$  (6; 22; 23) (Fig. 36).

First let us construct the multiview drawing (Fig. 36, a):

1. Draw a horizontal line which is taken as the ground line  $Ox$ .
2. From point  $O$  lay off coordinates  $x=20$  mm and  $x=6$  mm, given by the coordinates of points  $A$  and  $B$ .
3. Through the points  $A_{12}$  and  $B_{12}$  so obtained draw perpendiculars to line  $Ox$ .
4. On these perpendiculars from points  $A_{12}$  and  $B_{12}$ , below  $Ox$ ,

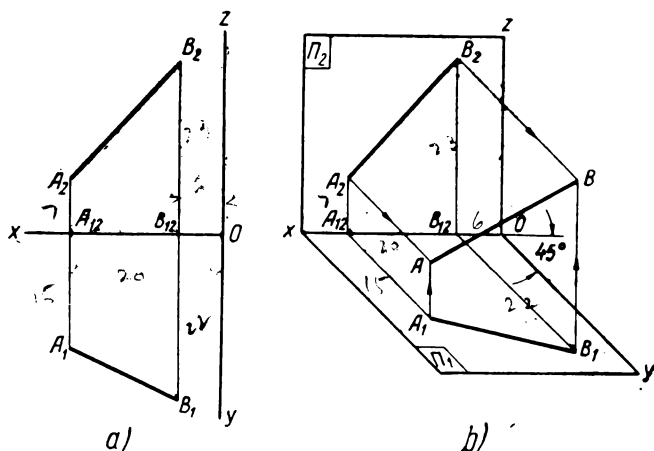


Fig. 36

lay off distances  $y$ , given by the coordinates of points  $A$  and  $B$ . These are respectively 15 and 22 mm, and above  $Ox$  lay off  $z=7$  mm and  $z=23$  mm. Note, the coordinates  $y$  and  $z$  are positive. The pairs of points  $A_1$  and  $B_1$  and  $A_2$  and  $B_2$  so obtained, are respectively the horizontal and frontal projections of points  $A$  and  $B$ .

5. By joining the corresponding projections of the points  $A$  and  $B$  the horizontal projection  $A_1B_1$  and the frontal projection  $A_2B_2$  of line  $AB$  are obtained.

In order to construct a space drawing of line  $AB$  (Fig. 36, b), in plane  $\Pi_1$  straight line  $A_1B_1$  is constructed using the coordinates  $x$  and  $y$  of points  $A$  and  $B$ . In plane  $\Pi_2$  line  $A_2B_2$  is constructed using the coordinates  $x$  and  $z$  of points  $A$  and  $B$ . To obtain the representation of the point  $A$  and  $B$  in space, projectors, respectively parallel to  $Oz$  and  $Oy$ , are drawn through points  $A_1$  and  $A_2$  and through points  $B_1$  and  $B_2$ . The intersection of the projectors passing through  $A_1$  and  $A_2$  gives the position of point  $A$  in space. The intersection of the projectors passed through  $B_1$  and  $B_2$  gives the position of point  $B$ . Joining points  $A$  and  $B$  we obtain the three-dimensional representation of line  $AB$ .

## § 7. The Projections and Positions of Straight Lines Relative to the Planes of Projection

1. An *oblique line* is inclined to all three planes of projection. In Fig. 37, *a* the representation in space of an oblique line is given and in Fig. 37, *b*, its multiview drawing.

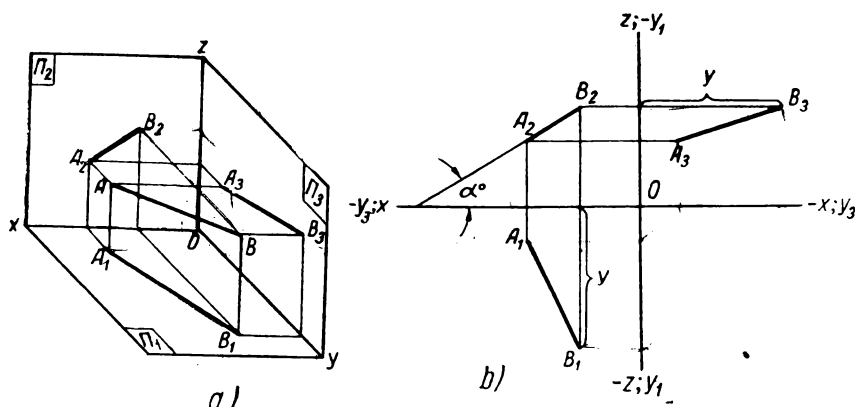


Fig. 37

On the multiview drawing it is seen that all three projections of the line are inclined to the ground lines, i. e., to the  $x$ ,  $y$  and  $z$  axes.

Each of the projections  $A_1B_1$ ,  $A_2B_2$  and  $A_3B_3$  of the straight line  $AB$  on the multiview drawing is shorter than  $AB$  itself. This can easily be proved. In Fig. 38 the line  $AB$ , its projection  $A_1B_1$  on plane  $\Pi_1$  and the projectors  $AA_1$  and  $BB_1$  are shown.

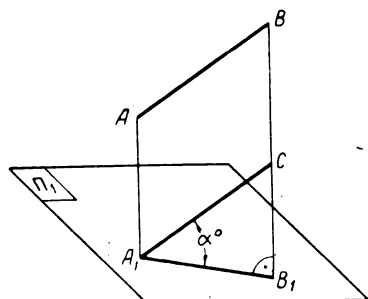


Fig. 38

Through point  $A_1$  draw a line  $A_1C_1$  parallel to  $AB$ . Angle  $A_1B_1C_1$  of the triangle obtained is a right angle ( $B_1B \perp \text{plane } \Pi_1$ ). If angle  $B_1A_1C_1$  is  $\alpha^\circ$ , then  $A_1B_1 = A_1C_1 \cos \alpha^\circ$ , but  $A_1C_1 = AB$ . Therefore  $A_1B_1 = AB \cos \alpha^\circ$ , and it follows that all projections of  $AB$  such as  $A_1B_1$  are shorter than the line  $AB$  itself. If  $\alpha^\circ = 0$ , then  $A_1B_1 = AB$ .

If  $\alpha^\circ = 90^\circ$ , then  $A_1B_1 = 0$  and line  $AB$  is projected onto plane  $\Pi_1$  as a point.

The angles  $\alpha$ ,  $\beta$  and  $\gamma$  formed by the projections of line  $AB$  with the ground lines  $x$ ,  $y$  and  $z$ , are not equal to the angle of inclination of the line itself to the planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$ .

For example, since the horizontal projecting plane  $ABB_1A_1$  (in which lines projecting  $AB$  onto the horizontal plane  $\Pi_1$  lie) is not parallel to the plane  $\Pi_2$ , the angle at which the line  $AB$  meets plane  $\Pi_1$  is not equal to the angle  $\alpha$  between the ground line  $Ox$  and projection  $A_2B_2$  (Fig. 37, b).

2. Lines in special positions. A straight line parallel to the horizontal projection plane  $\Pi_1$  is called a *horizontal line* or simply a

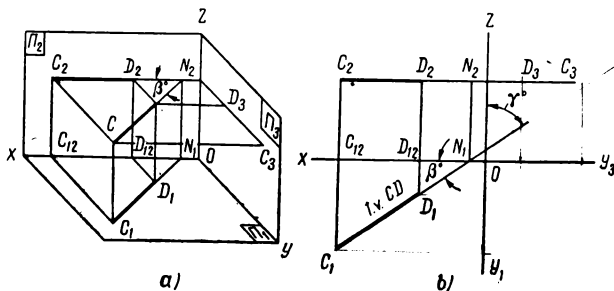


Fig. 39

horizontal (Fig. 39, a). Since all points of a horizontal lie at the same distance from the plane  $\Pi_1$ , the frontal projection of a horizontal is parallel to the ground line  $Ox$  ( $C_2D_2 \parallel Ox$ ). The profile projection  $C_3D_3$  is parallel to the line  $Oy_3$ . The horizontal projection of the horizontal line gives its true length, i. e.,  $C_1D_1 = CD$  (Fig. 39, b).

In the drawings the true length, or true value, is indicated by the letters *T. V.*

The angle between a straight line and a plane is the angle formed by the line itself and its orthographic projection on the given plane.

The angle  $\beta$  between the ground line  $Ox$  and the horizontal projection  $C_1D_1$  of the horizontal  $CD$  in the multiview drawing is equal to the angle of inclination of the horizontal to the frontal plane of projection  $\Pi_2$  and the angle  $\gamma$  between the line  $Oy_1$ , and the horizontal projection  $C_1D_1$  of the horizontal  $CD$  in the multiview drawing is the angle of inclination of the horizontal to the profile plane of projection  $\Pi_3$  (Fig. 39, b) and  $\angle \beta + \angle \gamma = 90^\circ$ .

3. A line parallel to the frontal plane of projection  $\Pi_2$  is called a *frontal line* or simply a *frontal* (Fig. 40). Since all points of a frontal lie at the same distance from plane  $\Pi_2$ , the horizontal projection  $E_1F_1$  of the frontal is parallel to the line  $Ox$ , the profile projection  $E_3F_3$  is parallel to the line  $Oz$ , and the frontal projection is equal in length to the projected frontal ( $E_2F_2 = EF$ , Fig. 40).

Angle  $\alpha$  between the ground line  $Ox$  and the frontal projection  $E_2F_2$  of the frontal (Fig. 40, *b*) gives the true inclination of the frontal to the horizontal plane of projection  $\Pi_1$ . The angle  $\gamma$  between the line  $Oz$  and frontal projection  $E_2F_2$  of the frontal gives the true inclination of the frontal to the profile plane of projection  $\Pi_3$  and  $\angle \alpha^\circ + \angle \gamma^\circ = 90^\circ$ .

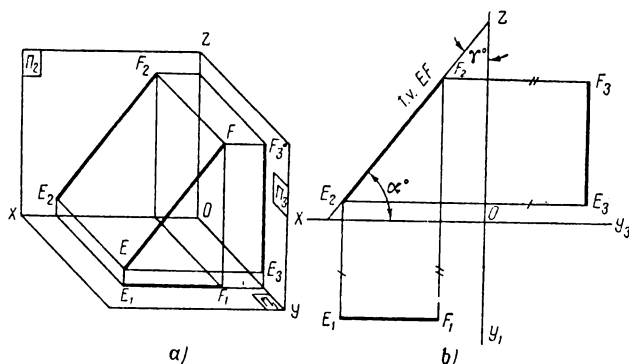


Fig. 40

4. A line parallel to the profile plane of projection  $\Pi_3$  is called a *profile line* (Fig. 41).

Since all points of a profile line lie at the same distance from the plane  $\Pi_3$  the horizontal projection  $P_1R_1$  and the frontal projection  $P_2R_2$  of the profile line are perpendicular to the ground line  $Ox$ , and the length

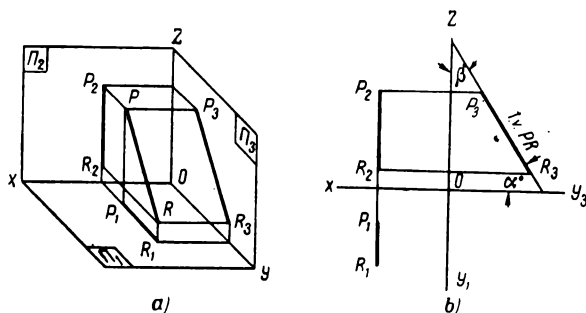


Fig. 41

of the profile projection  $P_3R_3$  is the true length of the line  $PR$  (Fig. 41, *b*).

The angles  $\alpha$  and  $\beta$  (Fig. 41, *b*) give the true values of the angles of inclinations of the line  $PR$  to planes  $\Pi_1$  and  $\Pi_2$ , respectively. The sum of the angles  $\alpha$  and  $\beta$  is  $90^\circ$ .

The profile line, shown in Fig. 41, is called an *ascending line* and that, shown in Fig. 42, a *descending line*.

The side pieces of a ladder leaning against a wall illustrate the former. The sides of a picture frame hanging at an angle on the wall illustrate the latter.

5. Lines parallel to two of the three planes of projection. Any line parallel to two planes of projection is obviously perpendicular to the third plane of projection and therefore is projected onto it as a point. In Fig. 43 lines  $AB$ ,  $CD$  and  $EF$  are shown.  $AB$  is parallel to the planes of projection  $\Pi_2$  and  $\Pi_3$ , therefore the length of this line when projected onto planes  $\Pi_2$  and  $\Pi_3$  is the true length of  $AB$ , and its projection on plane  $\Pi_1$  is a point. Line  $CD$  is parallel to the planes  $\Pi_1$  and  $\Pi_3$ , so its projections on those planes give the true length of the line. Its projection on plane  $\Pi_2$  is a point.  $EF$  is parallel to planes  $\Pi_1$  and  $\Pi_2$ , and its projection on those planes gives the true length of  $EF$  itself. On plane  $\Pi_3$  its projection is a point.

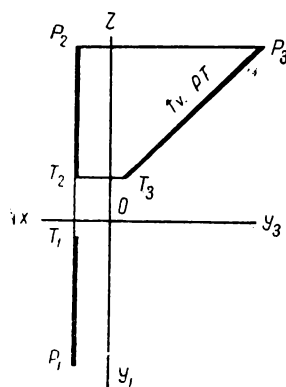


Fig. 42

6. Lines lying in the planes of projection. In Fig. 44 is shown a line  $AB$ , lying in the plane  $\Pi_1$ . Its horizontal projection  $A_1B_1$  coincides with the line itself ( $A_1B_1 \equiv AB$ ).

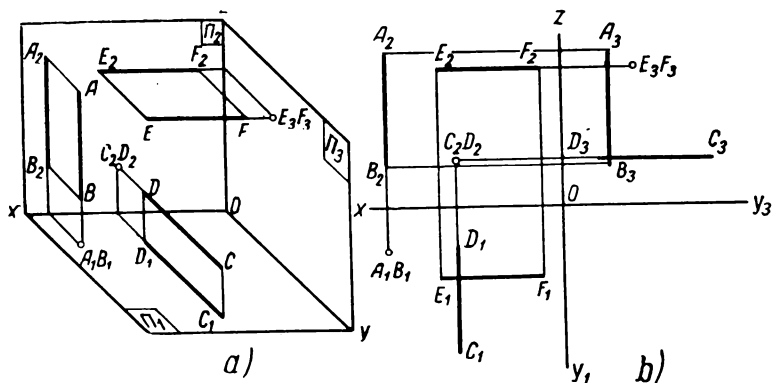


Fig. 43

The frontal projection  $A_2B_2$  lies on the  $Ox$  ground line and the profile projection  $A_3B_3$  lies on the  $Oy$  ground line.

In Fig. 45 the line  $CD$  lies in the plane  $\Pi_2$ . Its frontal projection



$C_2D_2$  coincides with the line itself ( $C_2D_2 \equiv CD$ ), the horizontal projection  $C_1D_1$  is situated on the ground line  $Ox$  and the profile projection  $C_3D_3$  lies on the ground line  $Oz$ .

If a line lies in the  $\Pi_3$  plane, its profile projection coincides with the line itself, the horizontal projection lies on the line  $Oy$  and the frontal projection, on the line  $Oz$ .

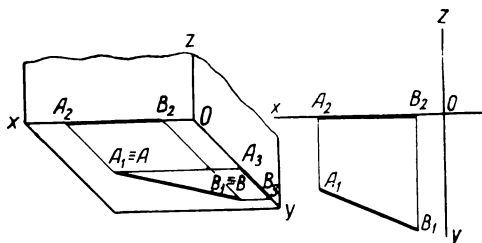


Fig. 44

7. Straight lines of limited length situated in the first quadrant or in the first octant have already been examined above. If a line lies in any other quadrant or octant, the construction of two or three of its projections consists in obtaining the orthographic projections of the points limiting the line and joining up the corresponding pairs of projection. The following three-dimensional and multiview drawings serve as examples:

1. Line  $EF$  is a horizontal located in the second quadrant (Fig. 46).
2. Line  $KL$  is a frontal located in the third quadrant (Fig. 47).
3. Line  $MN$  is an oblique line located in the fourth quadrant (Fig. 48).

#### Problems and Exercises

1. When is the projection of a straight line itself a straight line, and when does it become a point?
2. How many projections of a line are required to determine its position in space? Why is one projection insufficient for this purpose?
3. How are the projections of a horizontal located?
4. How are the projections of a frontal located?
5. What is the position of a line in space if its two principal projections, i. e., on  $\Pi_1$  and  $\Pi_2$ , are perpendicular to the ground line  $Ox$ ?
6. Describe the projections on planes  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  of a horizontal which is perpendicular to plane  $\Pi_2$ .
7. Describe the projections of a line which lies in the planes  $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$ .
8. In which space quadrants are the lines  $AB$ ,  $CD$  and  $EF$  situated

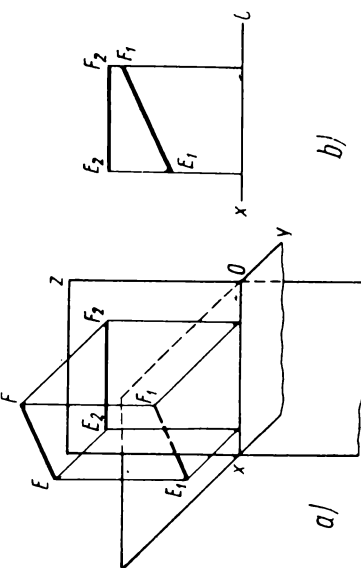


Fig. 45

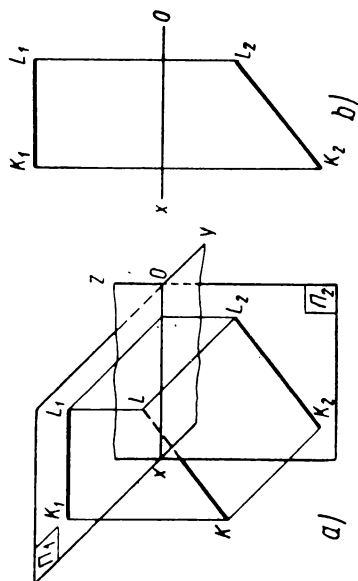


Fig. 47

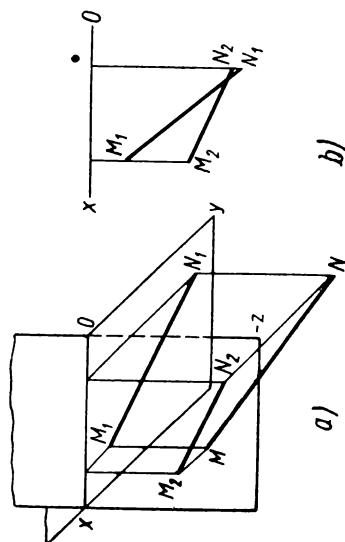


Fig. 46

Fig. 48

if their coordinates are  $A (13; -5; 10)$ ,  $B (5; -9; 10)$ ,  $C (5; -10, 20)$ ,  $D (18; -16; 4)$ ,  $E (14; 4; -6)$ ,  $F (18; 8; -9)$ ?

*Hint.* In this case two planes of projections  $\Pi_1$  and  $\Pi_2$  should be visualized. Having determined the quadrant in which the limiting points of the line lie, imagine them joined by a straight line.

9. Construct the multiview drawing, consisting of three projections, and the space drawing of line  $AB$  by means of two given

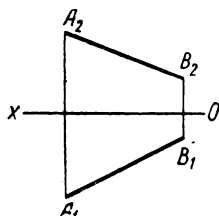


Fig. 49

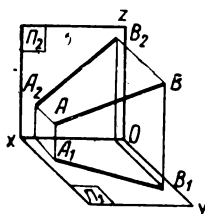


Fig. 50

projections and determine the coordinates of the ends of the given line (Fig. 49).

10. Construct the multiview drawing, consisting of three projections of the line  $AB$  using the space drawing. Determine the coordinates of the end points of the line (Fig. 50).

11. Construct the space drawing and the multiview drawing, two projections, of line  $AB$  using the coordinates of its limiting points  $A (20; 30; 40)$  and  $B (5; 5; 10)$ .

## § 8. The True Length of a Line and Its Angle of Inclination to the Planes of Projection\*

1. The line  $AB$  (Fig. 51) may be considered as the hypotenuse of a right triangle  $ABC$  in space, the right angle  $ACB$  is formed by the projector  $AA_1$  and line  $BC$  which is drawn parallel to the horizontal projection  $A_1B_1$  of the line  $AB$ . One of the sides ( $BC$ ) of the triangle  $ABC$  is equal to the horizontal projection  $A_1B_1$ , since parts of parallel lines limited by other parallel lines are equal in length. The length of the second side ( $AC$ ) is equal to the difference  $z - z'$  determined from the coordinates of points  $A$  and  $B$ .

On this basis the construction of the true length of  $AB$  in the multiview drawing (Fig. 51, *b*) is carried out as follows. Taking the horizontal projection  $A_1B_1$  as one side of a right triangle, through point  $A_1$  draw a perpendicular to  $A_1B_1$ , lay off on it from point  $A_1$  the length  $A_1\bar{A}$  equal to the difference coordinates  $z$  and

\* The method of finding the true length of a line by means of conversion of its projections is given in Chapter VI.

$z'$ , using the given coordinates of points  $A$  and  $B$ . The point  $\bar{A}$  obtained is joined to point  $B_1$ . The hypotenuse  $\bar{A}B_1$  of the right triangle constructed gives the true length of  $AB$ .

The true length of a line may be determined by constructing a right triangle if the frontal projection of the line serves as the first side. In this case the length of the second side will be equal to

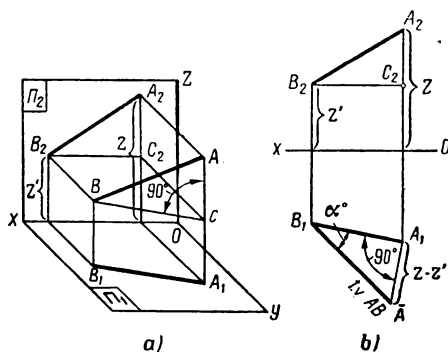


Fig. 51

the difference  $y - y'$  determined by means of the given coordinates of  $A$  and  $B$ . This construction is shown in Fig. 55.

Thus the true length of a line in the multiview drawing is constructed as the hypotenuse of a right triangle, one side of which is equal in length to one of the projections of the line and the second side of which is equal to the difference between the distances of the extreme points of the line from the plane of projection.

2. The true angle of inclination of a line to the plane of projection may also be determined by constructing a right triangle. In Fig. 51,  $a$  angle  $ABC$  is the angle between line  $AB$  and plane  $\Pi_1$ . Its true value is given in Fig. 51,  $b$  by angle  $A_1B_1\bar{A}$ , i. e., by the acute angle  $\alpha$ , formed by the horizontal projection  $A_1B_1$ , and the hypotenuse of the right triangle  $A_1B_1\bar{A}$ , with the help of which the true length of line  $AB$  was found.

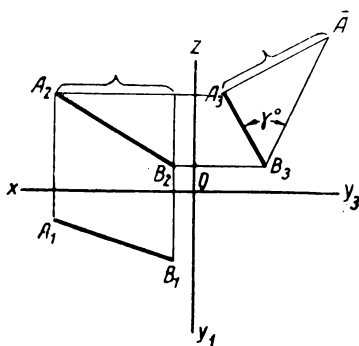


Fig. 52

The true values of angles  $\beta$  and  $\gamma$ , i. e., the angles of inclination of line  $AB$  to the planes of projection  $\Pi_2$  and  $\Pi_3$  respectively, can be determined by constructing right triangles. To find  $\beta^\circ$ , the angle of inclination of  $AB$  to  $\Pi_2$ , the first side should be the frontal projection. To find  $\gamma$ , the angle of inclination of  $AB$  to  $\Pi_3$ , the side should be the profile projection.

**Example.** It is required to determine  $\gamma^\circ$ , the angle of inclination of line  $AB$  to the profile plane of projection  $\Pi_3$  (Fig. 52).

Construct the right triangle  $A_3B_3\bar{A}$ , the first side of which will be the profile projection of line  $AB$ . The second side will be given by the difference between the distances of the extreme points of  $A_2$  and  $B_2$ . The angle  $\gamma$  formed between projection  $A_3B_3$  and hypotenuse  $B_3\bar{A}$  is the required angle.

### § 9. Relative Position of Points and Lines

If a point in space lies on a line, the projections of that point also lie on the corresponding projections of the line. To demonstrate this let there be given the line  $AB$ , its projections  $A_1B_1$  and  $A_2B_2$  and a point  $C$  on line  $AB$  (Fig. 53). To obtain the projections of point  $C$  draw projectors  $CC_1$  and  $CC_2$ . The projectors lie respectively in planes  $\alpha$  and  $\beta$ , which project line  $AB$  onto planes  $\Pi_1$  and  $\Pi_2$ . It follows that the projectors pierce the planes of projection  $\Pi_1$  and  $\Pi_2$  in points  $C_1$  and  $C_2$ , and these points lie on the corresponding lines of intersection of planes  $\alpha$  and  $\beta$  with the planes of projection  $\Pi_1$  and  $\Pi_2$ , in other words they lie on the projections  $A_1B_1$  and  $A_2B_2$ , which is what had to be proved.

If one projection of the point lies on the corresponding projection of a line, while another projection of the point does not lie on the corresponding projection of the line, then the point does not

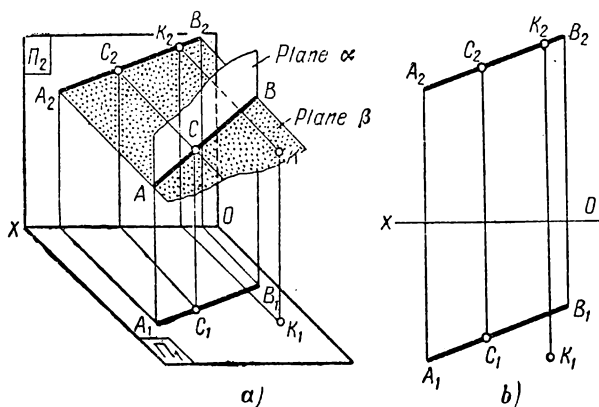


Fig. 53

lie on the given line in space. Such a case is also shown in Fig. 53, where the frontal projection  $K_2$  of point  $K$  lies on the frontal projection  $A_2B_2$  of line  $AB$ , while the point  $K$  itself does not lie on the line  $AB$ , since the horizontal projection  $K_1$  of point  $K$  does not lie on the horizontal projection  $A_1B_1$  of the line  $AB$ .

**Example.** It is required to mark on the given line  $AB$  a point  $C$  at an altitude of 15 mm from plane  $\Pi_1$  (Fig. 54).

At a distance of 15 mm from the ground line  $Ox$  draw a line parallel to it to intersect the frontal projection  $A_2B_2$  of line  $AB$ .

The point of intersection  $C_2$  is the frontal projection of the required point  $C$ . The horizontal projection  $C_1$  of point  $C$  is given by the intersection of a projector drawn through  $C_2$ , perpendicular to the ground line  $Ox$ , with the horizontal projection  $A_1B_1$  of line  $AB$ .

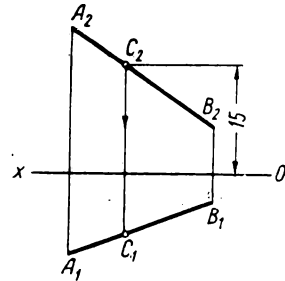


Fig. 54

## § 10. The Division of a Line in a Given Ratio

If a line in space is divided by a point in the ratio  $m:n$ , the projections of this line are also divided by the corresponding projections of that point in that same ratio  $m:n$ .

In the triangle  $C_2D_2\bar{D}$  (Fig. 55, *b*), with the help of which the true length  $C_2\bar{D}$  of line  $CD$  has been found, draw a straight line  $K_2\bar{K}$  parallel to the side  $D_2\bar{D}$ . This line  $K_2\bar{K}$  cuts off a triangle  $C_2K_2\bar{K}$ . This triangle is similar to triangle  $C_2D_2\bar{D}$ , and it follows

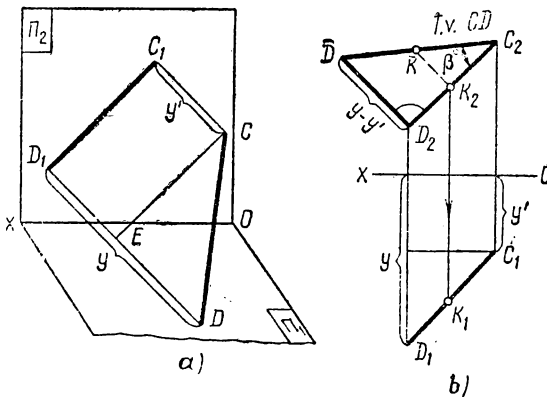


Fig. 55

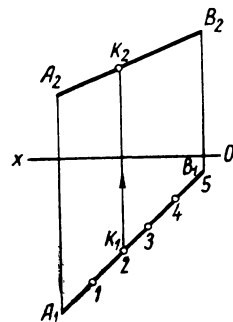


Fig. 56

that  $D_2K_2:K_2C_2=\overline{DK}:\overline{KC}_2$ ; in other words, the projection  $K_2$  of the point  $K$  divides the projection  $C_2D_2$  of the line  $CD$  in the same ratio as point  $K$  divides line  $CD$ .

**Example.** It is required to divide line  $AB$  in the ratio 2:3 (Fig. 56).

Divide one of the projections, for example  $A_1B_1$ , into five equal parts. From point  $A_1$  step off two parts, thus obtaining point  $K_1$  and through  $K_1$  perpendicular to the ground line  $Ox$ , draw a projector to cut  $A_2B_2$  in point  $K_2$ . The points  $K_1$  and  $K_2$  are projections of point  $K$  dividing the line  $AB$  in the ratio 2:3 as was required.

## § 11. Traces of Lines

The point in which a line pierces a plane of projections is called a *trace of that line*.

In a system containing three planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  an oblique line has three traces — the horizontal, frontal and profile. A line parallel to one of the planes of projection has two traces, and a line parallel to two planes of projection has one trace.

In Fig. 57 an oblique line  $AB$  is shown. It pierces plane  $\Pi_1$  in point  $M$  and plane  $\Pi_2$  in point  $N$ . Point  $M$  is therefore the horizontal trace of the line  $AB$  and point  $N$ , its frontal trace.

In the multiview drawing (Fig. 57, *b*) the horizontal trace of line  $AB$  can be found by determining the projection of the point contained both in the given line and in plane  $\Pi_1$ . In the frontal projection plane  $\Pi_2$  the projection of such a point is the point  $M_2$ , in which the frontal projection  $A_2B_2$  of the line  $AB$  intersects the ground line  $Ox$  on which lie the frontal projections of all points contained in plane  $\Pi_1$ , including point  $M$ . The horizontal projection  $M_1$  must lie on the projector drawn from point  $M_2$  perpendicular to the ground line  $Ox$ . It must, at the same time, lie on the projec-

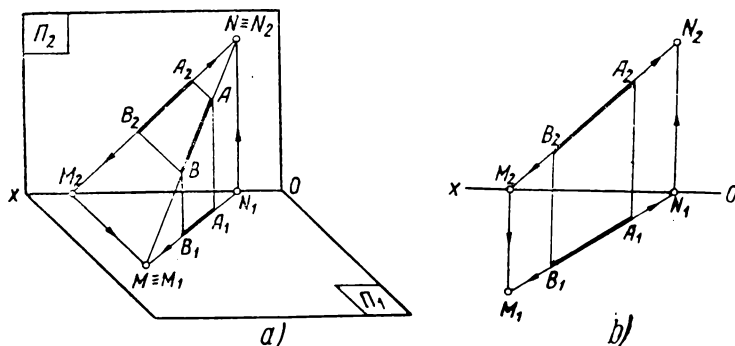
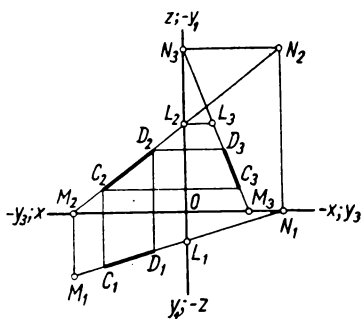


Fig. 57

1. Extend the frontal projection  $A_2B_2$  to the ground line  $Ox$ . The point  $M_2$  obtained is the frontal projection of the required trace  $M$ .



A perpendicular to the line  $Oz$  drawn through point  $L_2$  cuts the profile projection  $C_3D_3$  of the line in point  $L_3$  which is the profile trace of line  $CD$  and coincides with the profile trace itself ( $L_3 \equiv L$ ).



The horizontal projection  $L_1$  of the profile trace is the point of intersection of the horizontal projection  $C_1D_1$  of the line with the ground line  $Oy_1$ .

Attention should be paid to the fact that two of the three projections of each trace are situated on the respective ground lines ( $Ox$ ,  $Oy$ ,  $Oz$ ), while the third coincides with the trace itself.

The traces  $M$ ,  $N$  and  $L$  divide the line into parts situated in different octants. If the line  $CD$ , the projections of which are shown in Fig. 58, is considered as moving from left to right, it passes consecutively through the fourth, first, fifth and sixth octants.

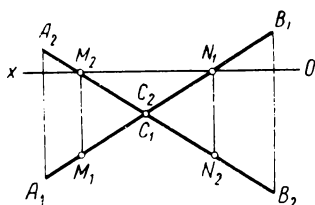


Fig. 60

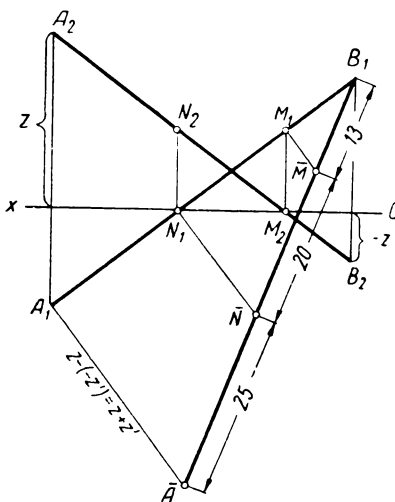


Fig. 61

In Fig. 59 the space drawing of line  $CD$  is shown with its traces  $M$ ,  $N$  and  $L$ .

Example 1. Determine through which quadrants line  $AB$  passes (Fig. 60).

Find the traces of line  $AB$ . The intersection of the frontal projection  $A_2B_2$  with the ground line  $Ox$  gives the frontal projection  $M_2$  of the horizontal trace  $M$ . By drawing a perpendicular from point  $M_2$  to the ground line  $Ox$  to intersect  $A_1B_1$ , point  $M_1$ , the horizontal projection of the trace, is obtained. It coincides with the trace  $M$  itself. Point  $N_1$ , where the horizontal projection  $A_1B_1$  cuts the ground line  $Ox$ , is the horizontal projection of the frontal trace. A perpendicular to the ground line  $Ox$  from point  $N_1$  intersects  $A_2B_2$  in point  $N_2$ , which is frontal projection of the frontal trace of line  $AB$  and coincides with the frontal trace  $N$  itself.

Point  $A$  lies in the first quadrant, since the projection  $A_1$  lies under the ground line  $Ox$  and the projection  $A_2$  lies above that line. Point  $B$  lies in the third quadrant. The intermediate point  $C$  of line  $AB$  is located in the fourth quadrant, since its projections  $C_1$  and  $C_2$  lie below the ground line  $Ox$ . The part  $AM$  of line  $AB$  is located in the first quadrant, part  $MN$  lies in the fourth and part

$AB$  — in the third quadrant. So line  $AB$  passes through the first, fourth and third space quadrants.

**Example 2.** Determine the true length of the parts of line  $AB$ , located in the various quadrants (Fig. 61).

The true lengths of  $AB$  and its parts will be determined by the right triangle method. Let the projection  $A_1B_1$  be the first side of the triangle. From point  $A_1$  draw a perpendicular to  $A_1B_1$  and from point  $A_1$  lay off on it a distance equal to the difference between the distances of points  $A_2$  and  $B_2$  from  $Ox$ , i. e.,  $z - (-z') = z + z'$ . Point  $\bar{A}$  thus obtained is joined to point  $B_1$ . The length of  $\bar{A}B_1$  is the true length of line  $AB$ .

To determine the lengths of the parts of line  $AB$  lying in the different space quadrants, through the horizontal projections  $M_1$  and  $N_1$  of the traces  $M$  and  $N$  draw perpendiculars to the horizontal projection  $A_1B_1$ . These perpendiculars intersect line  $\bar{A}B_1$  in points  $\bar{M}$  and  $\bar{N}$ , respectively, and divide  $\bar{A}B_1$  in the same ratio as points  $M_1$  and  $N_1$  divide the projection  $A_1B_1$  of line  $AB$ . Measuring the distance between points  $\bar{A}$ ,  $\bar{N}$ ,  $\bar{M}$  and  $B_1$  the length of the part  $AN$  lying in the first quadrant is found to be 25 mm. The length of part  $NM$  in the second quadrant is 20 mm and, finally, the length of part  $MB$  located in the third space quadrant is 13 mm.

#### Problems and Exercises

1. How is the true length of a line determined by means of the right triangle method?

2. How can a line given in the multiview drawing by two projections be divided in the ratio  $m : n$ ?

3. How are the horizontal and profile traces of a line constructed?

4. Line  $AB$  is given by the coordinates of its extremities  $A$  (15; 15; 12) and  $B$  (5; 8; 4): a) find the true length of  $AB$  and the angles of inclination of line  $AB$  to planes  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$ , i. e., angles  $\alpha$ ,  $\beta$  and  $\gamma$ ; b) indicate through which octants line  $AB$  passes.

**Hint.** To answer this question (b) construct the traces  $M$ ,  $N$  and  $L$  of the assigned line  $AB$ .

5. Through point  $A$  (5; 8; 6) draw a line, the traces of which on planes  $\Pi_1$  and  $\Pi_2$  lie at the same distances from the ground line  $Ox$ .

**Hint.** The required line in the profile plane of projection  $\Pi_3$  will pass through point  $A_3$  making an angle of  $45^\circ$  with the axes  $Oz$  and  $Oy_3$ .

### § 12. Projections of Parallel, Intersecting and Non-intersecting Lines

Two lines may occupy the following relative positions in space:

1) they may be parallel, 2) they may intersect, or 3) not intersect.

1. The corresponding projections of parallel lines are parallel to each other.

If through the given parallel lines  $AB$  and  $CD$  projecting planes are passed, they are parallel and their intersections with the plane of projection  $\Pi_1$  give two parallel lines  $A_1B_1$  and  $C_1D_1$ . These are the orthographic projections of the given lines  $AB$  and  $CD$  on plane

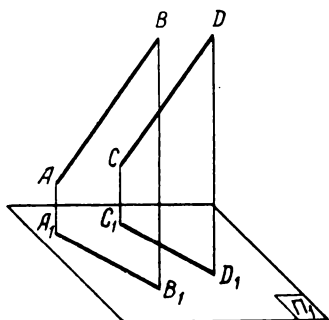


Fig. 62

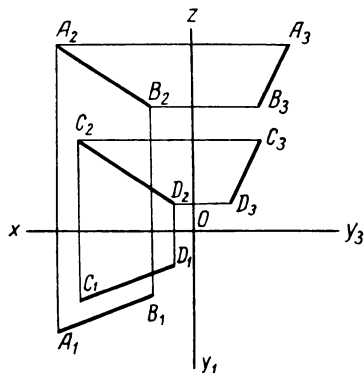


Fig. 63

$\Pi_1$  (Fig. 62). In the multiview drawing (Fig. 63) it can be seen that corresponding projections of parallel lines  $AB$  and  $CD$  are themselves parallel, i. e.,  $A_1B_1 \parallel C_1D_1$ ,  $A_2B_2 \parallel C_2D_2$  and  $A_3B_3 \parallel C_3D_3$ .

2. Intersecting lines, for example  $AB$  and  $CD$  (Fig. 64), have a common point  $K$ . Therefore the horizontal projection  $K_1$  and frontal

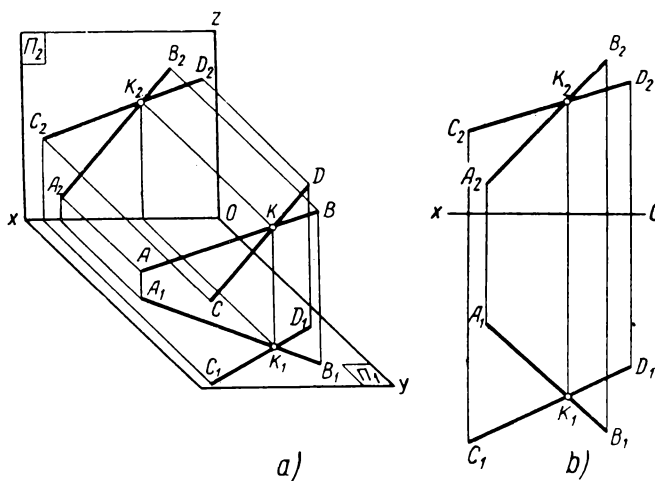


Fig. 64

projection  $K_2$  of this point must lie at the intersection of the corresponding projections of the given lines.

The points  $K_1$  and  $K_2$ , being projections of one and the same point  $K$  in space, in the multiview drawing (Fig. 64, *b*) are situated on one and the same perpendicular to the ground line.

This property of the intersecting lines is also true for a combi-

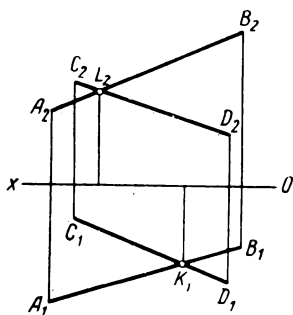


Fig. 65

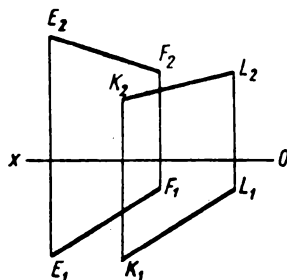


Fig. 66

nation of other projections, for example, their frontal and profile projections.

3. If two lines are not parallel and do not intersect, they are called *non-intersecting* or *skew*.

In the multiview drawing (Fig. 65) the points of intersection ( $K_1$  and  $L_2$ ) of corresponding projections of non-intersecting lines  $AB$  and  $CD$  ( $A_1B_1$  with  $C_1D_1$  and  $A_2B_2$  with  $C_2D_2$ ) do not lie on one perpendicular to the ground line  $Ox$ .

Through point  $A$  in space a line parallel to  $CD$  is passed and through point  $C$ , a line parallel to  $AB$ . To avoid overburdening the drawing, Fig. 65, these lines are not shown. The planes formed by the two pairs of mutually parallel lines are parallel to each other. These planes are called the *planes of parallelism* of non-intersecting lines.

If the planes of parallelism of two non-intersecting lines are perpendicular to one of the planes of projection ( $\Pi_1$ ,  $\Pi_2$  or  $\Pi_3$ ), then on that plane of projection the non-intersecting lines are projected as parallel lines (Fig. 66).

If the planes of parallelism of non-intersecting lines are parallel to one of the planes of projection ( $\Pi_1$ ,  $\Pi_2$  or  $\Pi_3$ ), then the non-intersecting lines are projected onto the remaining two planes of projection as parallel lines perpendicular to the ground line (Fig. 67).

**Example 1.** At a distance of 14 mm from plane  $\Pi_2$  draw a frontal to intersect two parallel lines  $AB$  and  $CD$  (Fig. 68).

Draw a line at a distance of 14 mm below the ground line  $Ox$  and parallel to it. This line intersects the horizontal projections  $A_1B_1$  and  $C_1D_1$  of the given lines in points  $K_1$  and  $L_1$ . The line  $K_1L_1$  is the horizontal projection of the frontal. The frontal projections of points  $K$  and  $L$  are found in the usual manner. These are found at the intersection of projections  $A_2B_2$  and  $C_2D_2$  with verticals

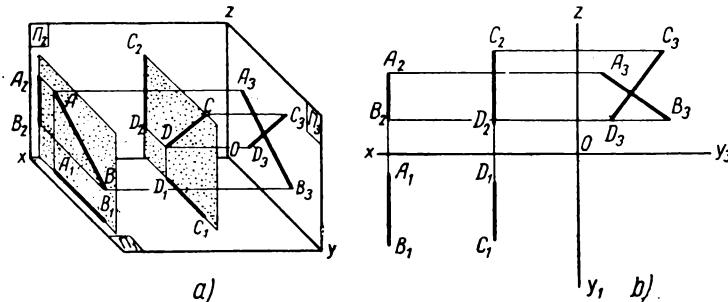


Fig. 67

passing through points  $K_1$  and  $L_1$ . Points  $K_2$  and  $L_2$  are then joined up. Line  $K_2L_2$  is the frontal projection of the frontal  $KL$ .

Example 2. Construct a parallelogram, one diagonal of which is line  $AC$  (Fig. 69).

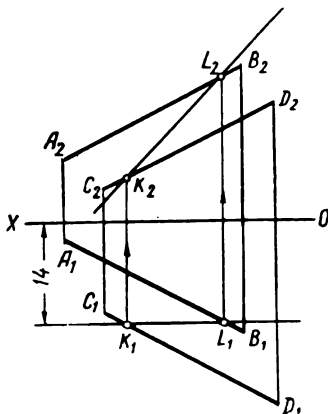


Fig. 68

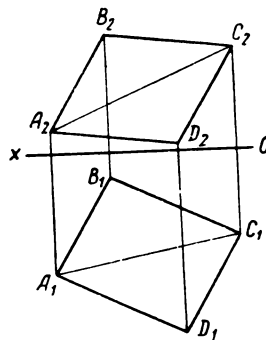


Fig. 69

Since in a parallelogram opposite sides are equal and parallel, we may take line  $AB$  of any length and direction as the side of the parallelogram. Then through point  $C$  pass line  $CD$  equal and parallel to line  $AB$ . Joining up corresponding projections of the pairs of points  $A$  and  $D$  and  $B$  and  $C$ , we obtain two projections of the parallelogram  $ABCD$ , which meet the stipulated condition.

### § 13. Projections of Plane Angles

1. A plane or linear angle formed by two straight lines, whether it is an acute, right or obtuse angle, is projected onto a plane in its true size, if both of its sides are parallel to that plane. This statement follows from the theorem of elementary geometry which states that angles formed by mutually parallel and similarly directed sides are equal.

In Fig. 70 the sides of the angle  $ABC$  are parallel to the plane of projection  $\Pi_1$ . Orthographic projections of lines which are parallel to the plane of projection  $\Pi_1$  are parallel to the lines themselves, since they are the opposite sides of rectangles formed by these lines, their projections and the projectors, i. e.,  $A_1B_1 \parallel AB$  and  $B_1C_1 \parallel BC$ . So the angles  $A_1B_1C_1$  and  $ABC$ , the corresponding sides of which are parallel, are equal.

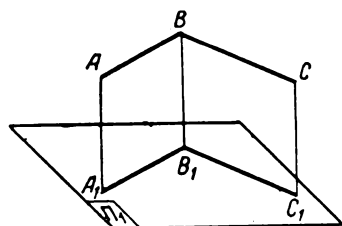


Fig. 70

Nevertheless it should be noted that to obtain the projection of an angle equal to the angle being projected, the parallel disposition of the sides of the latter relative to the plane of projection is a condition which, although sufficient, is not essential. It has been proved by Prof. N. Chetveroukhin in his theorem\* that it is possible

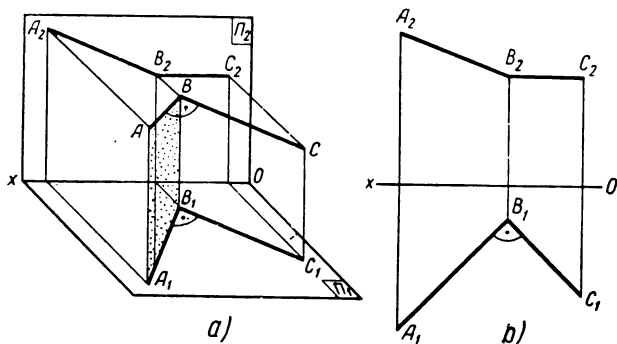


Fig. 71

for linear angles in space to occupy such positions that, while neither of their sides is parallel to the plane of projection, nevertheless the angles are projected orthographically onto the given plane in their true size. In such a case the inclination of the sides of the given angle to the plane of projection will necessarily be different.

\* "Any angle  $\alpha$  ( $0 < \alpha < 180^\circ$ ) may be considered as being the orthographic projection of any other angle  $\beta$  ( $0 < \beta < 180^\circ$ )."

2. If one of the sides of a right angle is parallel to the plane of projection, the right angle is projected onto the plane in its true size.

A right angle  $ABC$  in space is shown in Fig. 71, *a*. Its side  $BC$  is parallel to the horizontal plane of projection  $\Pi_1$  and its side  $AB$  is neither parallel nor perpendicular to that plane. In this case angle  $ABC$  is projected onto plane  $\Pi_1$  in its true size.

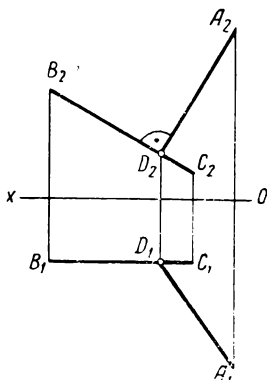


Fig. 72

This may be demonstrated as follows: the side  $BC$  is perpendicular to  $AB$  and to  $BB_1$ , since  $BB_1 \perp \Pi_1$ , and  $BC \parallel \Pi_1$ . Therefore  $BC$  is also perpendicular to the plane  $ABB_1A_1$  which in Fig. 71, *a* is shaded. The projection  $B_1C_1$  is parallel to  $BC$  itself. Therefore the projection  $B_1C_1$  is also perpendicular to the plane  $ABB_1A_1$ , and so, perpendicular to the line  $A_1B_1$  contained in that plane. In other words, angle  $A_1B_1C_1$  is a right angle.

From Fig. 71, *b* it may be seen that the right angle  $ABC$  is projected onto the horizontal plane  $\Pi_1$  without distortion, i. e., angle  $A_1B_1C_1 = 90^\circ$ , precisely because the side  $BC$  of the original angle is parallel to plane  $\Pi_1$ .

On the frontal plane of projection  $\Pi_2$  angle  $ABC$  is projected as angle  $A_2B_2C_2$  which is not equal to  $90^\circ$ . This is because neither of the sides of angle  $ABC$  is parallel to plane  $\Pi_2$ . Angle  $A_2B_2C_2$  may vary from  $0$  to  $180^\circ$ , depending on the position of its sides.

Example. From a point  $A$  draw a line perpendicular to the frontal  $BC$  (Fig. 72).

Since in this case the right angle will be projected onto plane  $\Pi_2$  in its true size, we draw from point  $A_2$  a perpendicular to  $B_2C_2$  and mark the point  $D_2$ , where it intersects with  $B_2C_2$ . Having

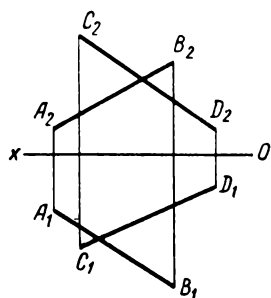


Fig. 73

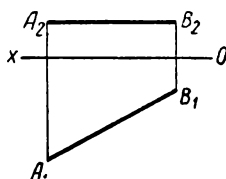


Fig. 74

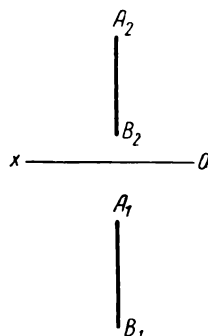


Fig. 75

found with the help of a projector point  $D_1$ , points  $A_1$  and  $D_1$  are then joined up. Then lines  $A_1D_1$  and  $A_2D_2$  are the projections of line  $AD$  which is perpendicular to  $BC$ .

### Problems and Exercises

1. Prove that corresponding projections of parallel lines are parallel to each other.

2. How are two intersecting lines shown in the multiview drawing?

3. How are two non-intersecting lines shown in the multiview drawing?

4. When is a plane right angle projected in true size: a) onto one plane of projection, b) onto two planes of projection?

5. Draw a horizontal to cut the non-intersecting lines  $AB$  and  $CD$  and determine the distance between the intersection points obtained (Fig. 73).

6. Construct a triangle with sides of different lengths and its base  $AB \parallel \Pi_1$  (Fig. 74).

7. Determine the distance of profile line  $AB$  from the ground line  $Ox$  (Fig. 75).

8. Determine the distance between the non-intersecting lines  $AB$  and  $CD$  (Fig. 76).

9. Construct a rectangle  $ABCD$  the side  $AB$  of which is a frontal line.

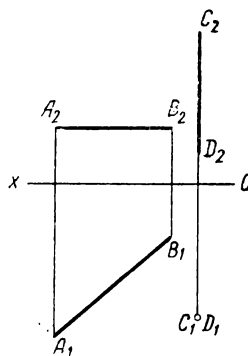


Fig. 76



## Chapter IV

### PLANES

#### § 14. Given Planes

1. In the multiview drawing a plane may be given by showing those geometrical elements by which the position of the plane in space is completely determined.

These elements are as follows:

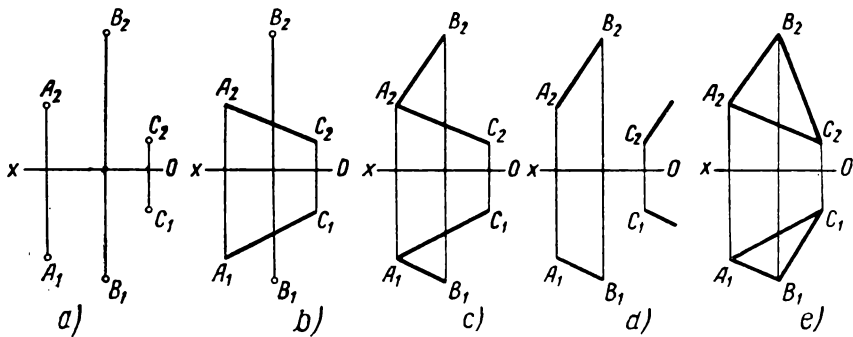


Fig. 77

1. Three points not situated on a straight line (Fig. 77, a).
2. A straight line and a point not contained in the straight line (Fig. 77, b).
3. Two intersecting lines (Fig. 77, c).
4. Two parallel lines (Fig. 77, d).
5. A triangle or any other plane figure (Fig. 77, e).

It is simple to change from one method of determining a plane to any other. It is sufficient, for instance, to join three given points by straight lines (Fig. 77, a) in order to obtain the representation of the planes of triangles (Fig. 77, e). Then again, for example, two points (of the three given) can be joined by a straight line and through the third point a line, parallel to the first, can be drawn. In this case instead of having a plane determined by three points,

we have it determined by two parallel straight lines (Fig. 77, d). Other similar changes may be made.

2. The elements determining a plane (Fig. 77) are sufficient to allow straight lines and points contained in the plane to be constructed.

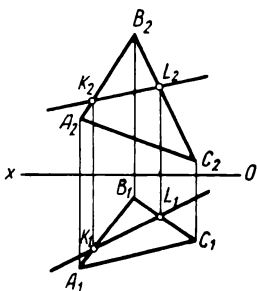


Fig. 78

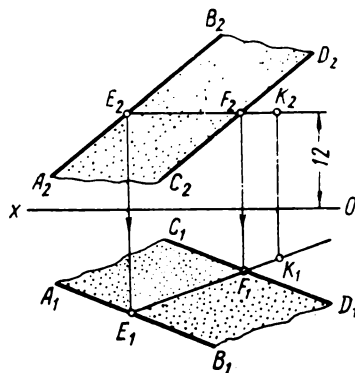


Fig. 79

For instance, if a plane is given in the multiview drawing by triangle  $ABC$  (Fig. 78), it is possible to construct the projections of any line, such as  $KL$ , lying in the given plane. A straight line is contained in a plane if two points of the line lie in that plane. Take two points  $K$  and  $L$  on the sides  $AB$  and  $BC$  of the triangle. In accordance with § 9 each projection of these points on corresponding projections of the sides  $AB$  and  $BC$  must lie on a line perpendicular to the ground line  $Ox$ , that is on a vertical projector. Then through the corresponding projections of points  $K$  and  $L$  the lines  $K_1L_1$  and  $K_2L_2$  are drawn. Line  $KL$  then lies in the plane of triangle  $ABC$ .

3. The construction of points contained in a given plane may be reduced in general to drawing an auxiliary line passing through a point and lying in a plane. For instance, if a plane is determined by the parallel lines  $AB$  and  $CD$  (Fig. 79), in order to construct a point  $K$  lying in the given plane and 12 mm from plane  $\Pi_1$ , first, at a distance of 12 mm from the ground line  $Ox$ , the frontal projection  $E_2F_2$  of an auxiliary horizontal  $EF$  is drawn. Next, with the help of projecting lines passing through points  $E_2$  and  $F_2$ , the points  $E_1$  and  $F_1$  are found, as shown by arrows. Thus the projection  $E_1F_1$  is also found. Then on the projections  $E_1F_1$  and  $E_2F_2$  of line  $EF$ , the projections  $K_1$  and  $K_2$  of point  $K$  are marked, taking into account that line  $K_1K_2$  must be perpendicular to the ground line  $Ox$ .

4. Traces of planes. When solving problems of descriptive geometry it is often convenient to determine a plane in the multiview drawing by giving its traces.

The *trace of a plane* is the straight line of intersection of that plane with the plane of projection.

A plane  $\alpha$  and its traces — the horizontal trace  $k$ , the frontal trace  $l$  and the profile trace  $m$  — are shown in Fig. 80, a.

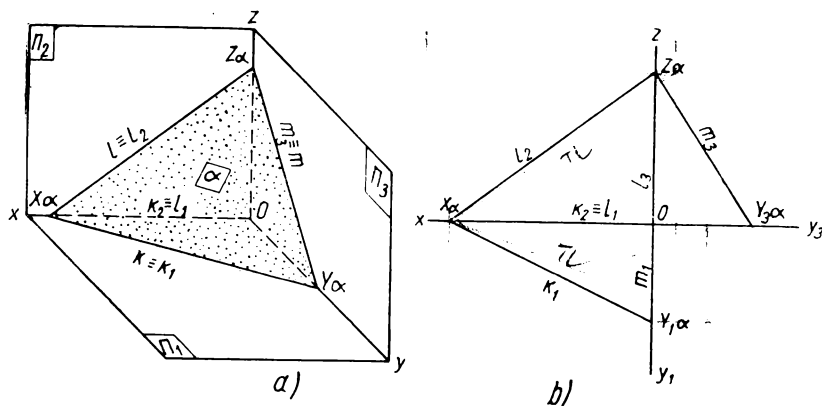


Fig. 80

The traces of planes coincide with the corresponding projections of those traces, i. e., trace  $k \equiv k_1$  which is its projection on  $\Pi_1$ , trace  $l \equiv l_2$  which is its projection on  $\Pi_2$ , and trace  $m \equiv m_3$  which is its projection on  $\Pi_3$ .

The other two projections of each trace coincide with the corresponding ground lines and, as a rule, they are not shown in the multiview drawing of a plane.

If a plane is inclined to all three planes of projection ( $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$ ), as illustrated in Fig. 80, a, then each pair of its traces intersect in a point. These are called the *points of convergence of the traces*.

The points of convergence  $X_\alpha$ ,  $Y_\alpha$ ,  $Z_\alpha$  of the traces are contained in the corresponding ground lines. This can be demonstrated as follows: trace  $k$ , being the line of intersection of planes  $\alpha$  and  $\Pi_1$ , is common to both of those planes. The trace  $l$ , being the line of intersection of planes  $\alpha$  and  $\Pi_2$ , is common to planes  $\alpha$  and  $\Pi_2$ . The point of intersection  $X_\alpha$  of traces  $k$  and  $l$  of plane  $\alpha$ , in other words, the point of convergence of the traces, must therefore belong to the three planes  $\Pi_1$ ,  $\Pi_2$  and  $\alpha$ . Moreover it follows that this point must lie on the ground line  $Ox$  common to planes  $\Pi_1$  and  $\Pi_2$ . Similarly it can be proved that the points of convergence  $Y_\alpha$  and  $Z_\alpha$  of the traces lie on the ground lines  $Oy$  and  $Oz$ , respectively.

## § 15. Oblique Planes

1. An *oblique plane* is inclined to all three planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$ , such as plane  $\alpha$ , shown in Fig. 80. Both in space and in the multiview drawing all the three traces

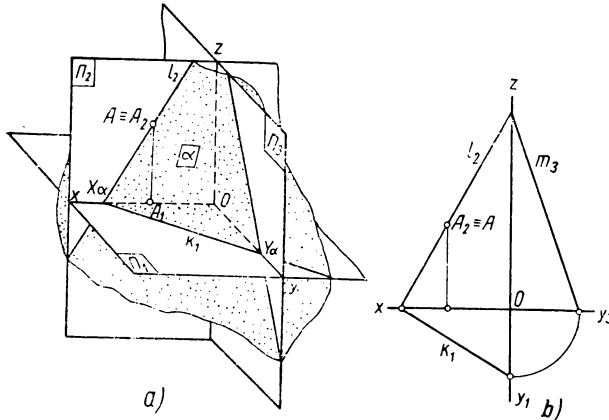


Fig. 81

$k$ ,  $l$  and  $m$  of plane  $\alpha$  are inclined to ground lines  $Ox$ ,  $Oy$  and  $Oz$  (Fig. 80,  $a$ ,  $b$ ). It should be noted that the angles formed by the traces of plane  $\alpha$  with the ground lines are not the same as the angles of inclination of plane  $\alpha$  itself to the planes of projection (see § 17).

It should also be noted that the traces of oblique planes contain the projections of only those of its points which actually lie on the traces, such as the point  $A$  which lies on the frontal trace  $l$  of plane  $\alpha$  in Fig. 81. No other points belonging to oblique planes can have their projections lying on any of the traces of the given planes. Projections of such points in oblique planes can be found by means of auxiliary lines contained in the planes, as shown in Fig. 79.

2. The upper part of the oblique plane  $\alpha$ , depicted in Fig. 81,  $a$ , slopes away from the reader. Each

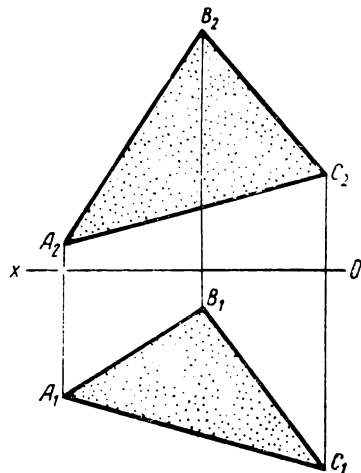


Fig. 82

of the three plane angles formed by the intersection of the traces of this plane in the first octant, is an acute angle. Even so, on the multiview drawing the angles between the traces are not necessarily acute angles. In Fig. 81, *b*, for example, the angle  $\alpha$  formed by the traces  $l$  and  $k$  of plane  $\alpha$  is actually an obtuse angle.

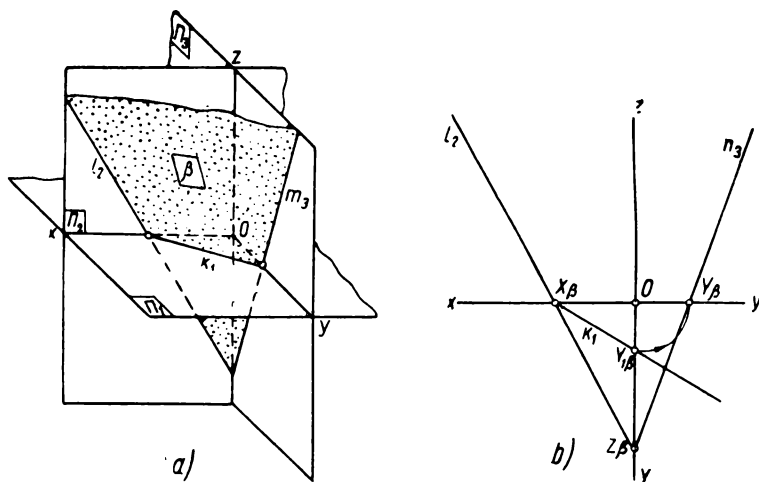


Fig. 83

3. The oblique plane in Fig. 82 is determined by the triangle  $ABC$ . The top part of this plane also slopes away from the reader. So, when the two projections on planes  $\Pi_1$  and  $\Pi_2$  are constructed the observer will see one and the same side of the plane. The projections of this side of the plane are indicated by dots in Fig. 82.

4. The top of plane  $\beta$  shown in Fig. 83, *a* slopes toward the observer. The angle between its horizontal and frontal traces, within the limits of the first octant, is an obtuse angle. The traces  $k$  and  $m$  of plane  $\beta$ , within the limits of the first octant, also form an obtuse angle.

The construction in the multiview drawing (Fig. 83, *b*) of profile trace  $m$  of this plane, when the horizontal  $k$  and frontal  $l$  traces are given, consists, as before, in finding the points of convergence  $Z_\beta$  and  $Y_\beta$  and drawing a straight line  $m_3$  through them.

5. The top of the oblique plane, determined by the triangle  $ABC$  in Fig. 84, is also inclined toward the observer. Therefore the side, seen when projecting the plane onto  $\Pi_2$  — the frontal projection of this side is indicated by hatching — is not seen when projecting the plane onto  $\Pi_1$ . The horizontal projection of the latter is indicated by dots.

6. The oblique plane  $\gamma$ , shown in Fig. 85, *a*, is characterized by the fact that in the multiview drawing (Fig. 85, *b*) its traces  $k_1$  and  $l_2$  merge as one straight line. In space these traces cannot constitute one straight line since they lie in different planes,  $\Pi_1$  and  $\Pi_2$ , and neither of them coincides with the ground line  $Ox$ .

The fact that in the multiview drawing (Fig. 85, *b*) the angles formed by the traces  $k$  and  $l$  with the ground line  $Ox$  are the same is because these angles are also equal in space.

7. The construction of traces of planes. In Fig. 86 the line  $AB$  shown is contained in plane  $\alpha$  and pierces the planes of projection  $\Pi_1$  and  $\Pi_2$  in points  $M$  and  $N$  which are its traces. These traces lie on the corresponding traces  $k$  and  $l$  of plane  $\alpha$ . Indeed, if it be assumed that the traces of the given line  $AB$

do not lie on the corresponding traces of plane  $\alpha$ , then two points,  $M$  and  $N$ , of straight line  $AB$ , which does lie in plane  $\alpha$ , do not lie in the plane; this is obviously impossible.

Therefore the traces of straight lines contained in a plane lie on the corresponding traces of that plane.

It follows that the construction of the traces of oblique planes, determined by a combination of points and lines not situated in the planes of projection, consists in the construction of the traces of two lines contained in the given plane, and then drawing through these points the corresponding traces of the planes.

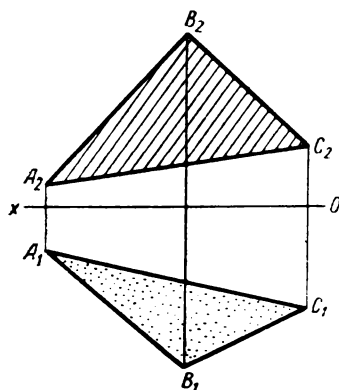


Fig. 84

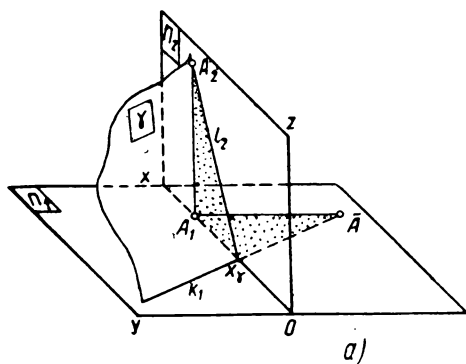
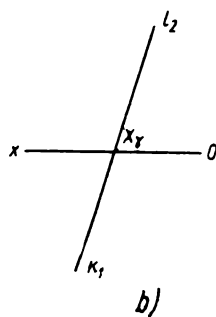


Fig. 85



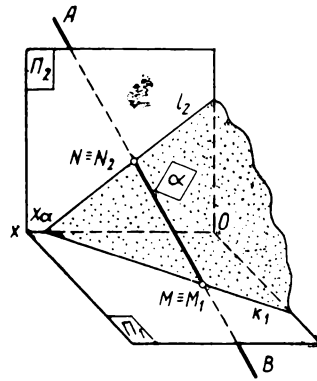


Fig. 86

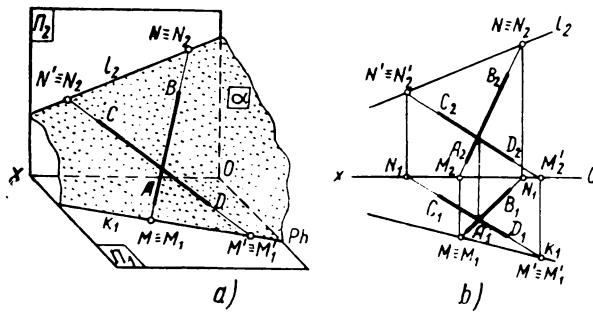


Fig. 87

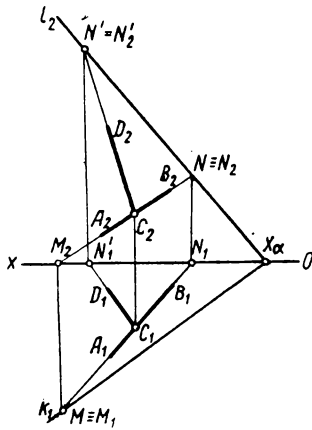


Fig. 88

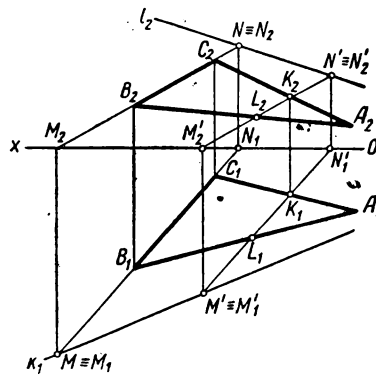


Fig. 89

In Fig. 87 the plane  $\alpha$  is given by two intersecting lines  $AB$  and  $CD$ . To construct its traces  $k$  and  $l$  on the multiview drawing (Fig. 87, *b*) the horizontal traces  $M$  and  $M'$  of the given lines are found (§ 11). Next, through  $M$  and  $M'$  a line  $k_1$  is passed. Then the frontal traces  $N$  and  $N'$  of the given lines are found, and the line  $l_2$  is drawn through them. The straight lines  $k_1$  and  $l_2$  are the horizontal and frontal traces of plane  $\alpha$ .

When the point  $X_\alpha$  of convergence of the traces of a plane is situated within the limits of the drawing, as in Fig. 88, to construct the traces of the plane given by intersecting lines, such as  $AB$  and  $CD$ , it is sufficient to find two traces of one line and only one trace of the other line. For instance, it is enough to find the traces  $M$  and  $N$  of the line  $AB$  and frontal trace  $N'$  of the line  $CD$ . The horizontal trace  $k_1$  of plane  $\alpha$  will pass through trace  $M$  and the point of convergence of the traces  $X_\alpha$ .

**Example.** To construct traces of the plane given by triangle  $ABC$  (Fig. 89).

In this particular case only one side of the triangle, namely  $BC$ , has traces within the limits of the drawing, whereas the sides  $AB$  and  $AC$  pierce the planes of projection  $\Pi_1$  and  $\Pi_2$  outside the drawing.

To solve the problem in the plane of triangle  $ABC$  an auxiliary line  $KL$  parallel to  $BC$  is drawn through any point  $K$  on the side  $AC$ . Once the traces  $M$  and  $N$  of line  $BC$  and traces  $M'$  and  $N'$  of line  $KL$  have been found, a horizontal trace  $k_1$  is drawn through points  $M$  and  $M'$ . Through points  $N$  and  $N'$  a frontal trace  $l_2$  is drawn. These are traces of plane  $\alpha$  given by the triangle  $ABC$ .

## § 16. Projecting Planes

1. A *horizontal-projecting plane* is a plane ( $\alpha$ ), perpendicular to the horizontal plane of projection  $\Pi_1$  (Fig. 90).

Since the plane  $\alpha$  is perpendicular to  $\Pi_1$ , the projector  $AA_1 \perp \Pi_1$ , drawn through point  $A$  of plane  $\alpha$ , also lies in plane  $\alpha$ . This projector will pierce plane  $\Pi_1$  in point  $A_1$  on the line of intersection of planes  $\alpha$  and  $\Pi_1$ , i. e., on the horizontal trace ( $k_1$ ) of plane  $\alpha$ . This trace coincides with line  $\alpha_1$  in which plane  $\alpha$  projects on  $\Pi_1$  ( $k_1 \equiv \alpha_1$ ).

All the projectors drawn perpendicular to plane  $\Pi_1$  through any points of plane  $\alpha$  pierce plane  $\Pi_1$  on that same trace  $k_1 \equiv \alpha_1$ .

Therefore the horizontal projections of all points contained in the horizontal-projecting plane ( $\alpha$ ) lie on the horizontal trace ( $k_1$ ) of that plane. This property of the horizontal trace of plane  $\alpha$  will be called the *collecting* property, and traces possessing this property will be called the *trace-projections of planes*.



Since planes  $\alpha$  and  $\Pi_2$  are perpendicular to one and the same plane  $\Pi_1$ , the line of their intersection (trace  $l$ ) is perpendicular to  $\Pi_1$ . This means that the trace  $l$  is also perpendicular to every other line on  $\Pi_1$  which passes through it, including the ground line  $Ox$ .

It follows that the frontal trace of the horizontal-projecting plane is perpendicular to the ground line  $Ox$ .

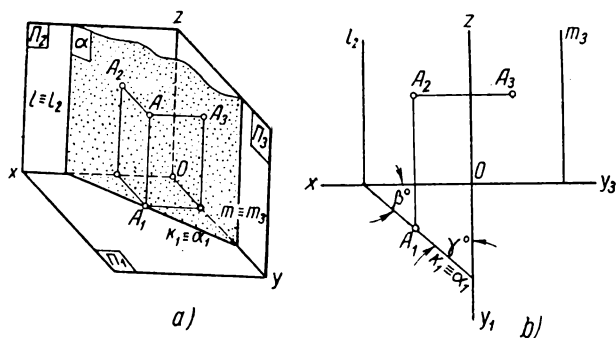


Fig. 90

In the same way the profile trace  $m_3$  of this plane both in space and in the multiview drawing (Fig. 90, b) is perpendicular to the ground line  $Oy_3$ .

Since the plane of projection  $\Pi_1$  is perpendicular to the frontal trace  $l$ , i. e., to the edge of the dihedral between planes  $\alpha$  and  $\Pi_2$ , the angle  $\beta$  formed by the horizontal trace-projection  $k_1 \equiv \alpha_1$  and the ground line  $Ox$ , is the true angle of inclination of plane  $\alpha$  to the frontal plane of projection  $\Pi_2$  (Fig. 90, b).

Since plane  $\Pi_1$  is also perpendicular to trace  $m$ , the angle  $\gamma$  between the trace-projection  $k_1 \equiv \alpha_1$  and ground line  $Oy_1$  gives the true angle of inclination of plane  $\alpha$  to the profile plane of projection  $\Pi_3$  (Fig. 90, b).

**Example.** It is required to pass a horizontal-projecting plane through line  $AB$  (Fig. 91).

Since the horizontal trace-projection  $\alpha_1$  of this plane possesses collecting properties, it should be drawn as a continuation of projection  $A_1B_1$ . The frontal trace  $l$  is perpendicular to ground line  $Ox$ .

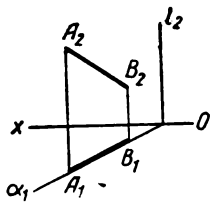


Fig. 91

2. The *frontal-projecting plane* is a plane perpendicular to the frontal plane of projection. In Fig. 92, a the frontal-projecting plane  $\beta$  is shown.

Its frontal trace-projection ( $l_2 \equiv \beta_2$ ) possesses collecting properties, i. e., the frontal projection

of any point of the plane lies on this trace-projection. An example of this is the frontal projection  $A_2$  of point  $A$ .

The other two traces,  $k_1$  and  $m_3$ , of the frontal-projecting plane are respectively perpendicular to ground lines  $Ox$  and  $Oz$  (Fig. 92 a, b).

Angles  $\alpha$  and  $\gamma$  are the true angles of inclinations of plane  $\beta$  to planes of projection  $\Pi_1$  and  $\Pi_3$  respectively.

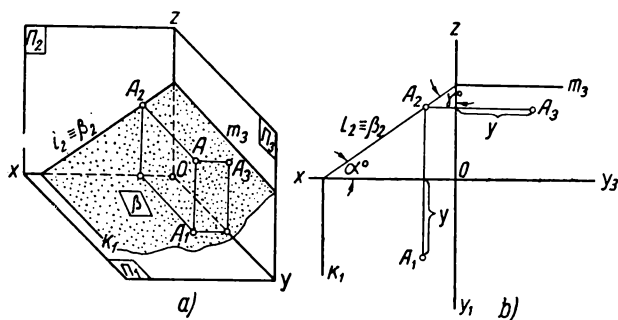


Fig. 92

**Example.** In Fig. 93 the frontal projection  $A_2B_2C_2$  of triangle  $ABC$  coincides with the frontal trace-projection  $\beta_2$  of plane  $\beta$ . This shows that triangle  $ABC$  is contained in plane  $\beta \perp \Pi_2$ .

3. The *profile-projecting plane* is a plane perpendicular to the profile plane of projection. In Fig. 94 the profile-projecting plane  $\gamma$  is shown.

Its profile trace-projection  $\gamma_3$  possesses collecting properties, i. e., the profile projection of any point belonging to plane  $\gamma$  lies on this trace-projection as does, for instance, the profile projection  $A_3$  of point  $A$ .

The remaining two traces,  $k_1$  and  $l_2$ , of the profile-projecting plane are respectively perpendicular to ground lines  $Oy_1$  and  $Oz$  (Fig. 94, b).

The angles  $\alpha$  and  $\beta$  give the true values, of the angles of inclination of the profile-projecting plane to the planes of projection  $\Pi_1$  and  $\Pi_2$ , respectively.

4. The *horizontal plane*  $\alpha$  (Fig. 95) is a special type of projecting plane. Since this plane, being parallel to plane  $\Pi_1$ , is at the same time perpendicular to the planes of projection  $\Pi_2$  and  $\Pi_3$ , its traces on  $\Pi_2$  and  $\Pi_3$  are perpendicular to ground line  $Oz$  and possess collecting properties.

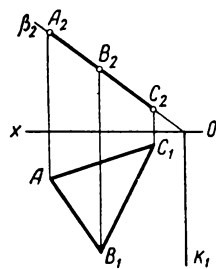


Fig. 93

So, on the trace-projections  $\alpha_2$  and  $\alpha_3$  the corresponding projections  $A_2$  and  $A_3$  of point  $A$  contained in the plane  $\alpha$  are to be found. Any plane figure contained in the horizontal plane is projected onto the plane of projection  $\Pi_1$  in its true size. For example, the regular pentagon  $ABCDE$  (Fig. 96), which lies in the horizontal plane, is projected onto plane  $\Pi_1$  without distortion. The frontal projection of this pentagon coincides with the trace-projection  $\alpha_2$  of plane  $\alpha$ .

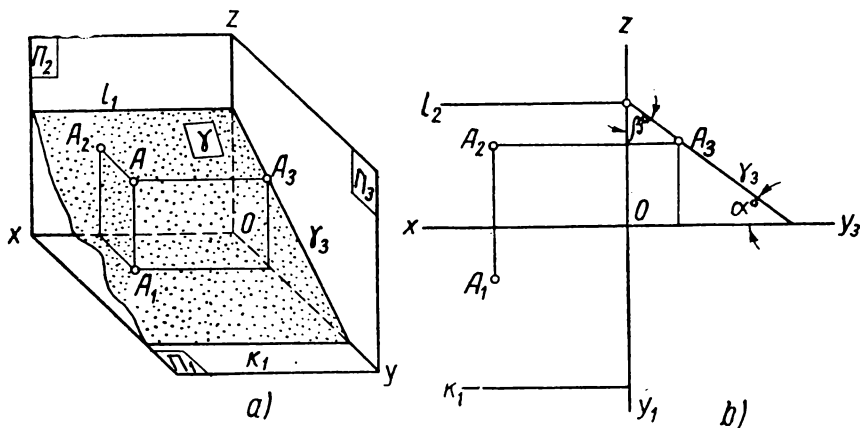


Fig. 94

5. The *frontal plane*  $\beta \parallel \Pi_2$  (Fig. 97) and the *profile plane*  $\gamma \parallel \Pi_3$  (Fig. 98) are also special cases of projecting planes. The trace-projections  $\beta_1$  and  $\beta_3$  of the frontal plane  $\beta$  are parallel to the respective ground lines  $Ox$  and  $Oz$ , and the trace-projections  $\gamma_1$  and  $\gamma_2$  of the profile plane  $\gamma$  on the multiview drawing merge into one

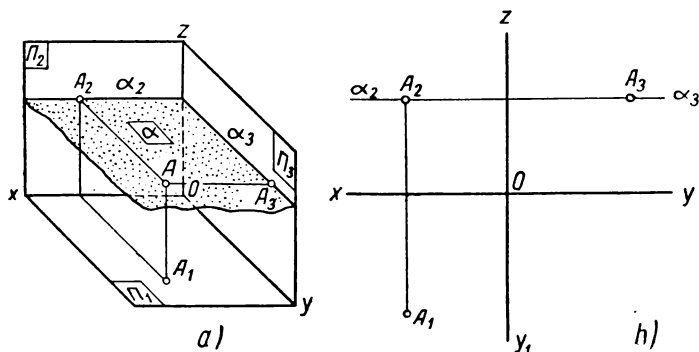


Fig. 95



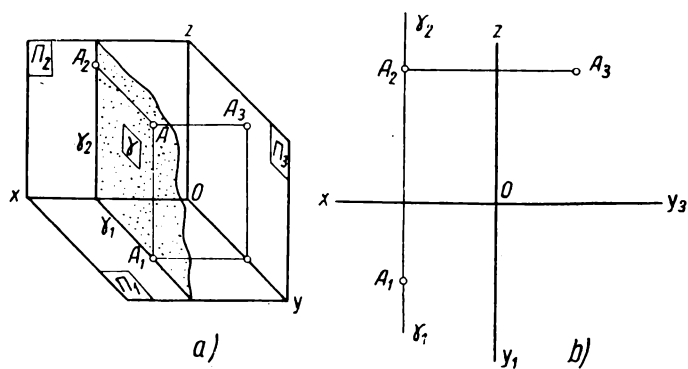


Fig. 98

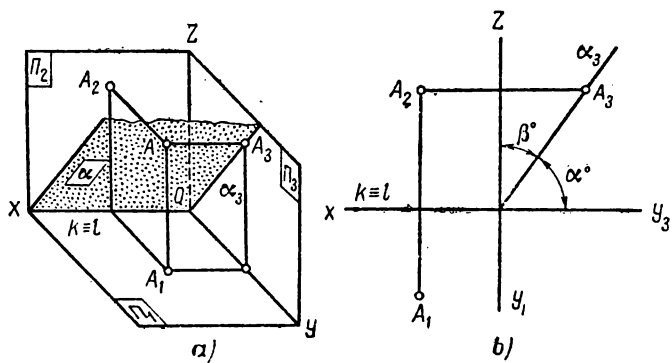


Fig. 99

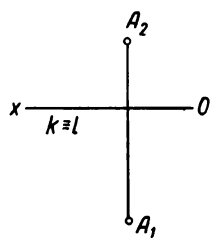


Fig. 100

If an axial plane has the same angle of inclination to two planes of projection, it is called a *bisecting plane*. In Fig. 102 a bisecting plane is depicted. It belongs to the first and third quadrants formed by the planes of projection  $\Pi_2$  and  $\Pi_1$ . This bisecting plane is the geometrical locus of points equidistant from the planes of projection  $\Pi_1$  and  $\Pi_2$ . The coordinates  $y$  and  $z$  of any point  $A$  assumed in this

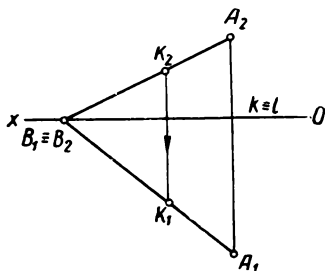


Fig. 101

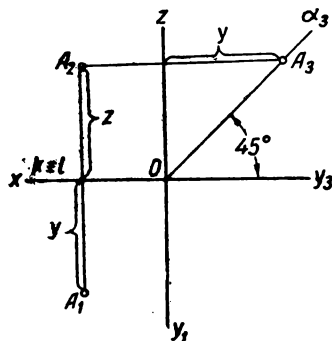


Fig. 102

plane are equal. Since the coordinates  $y$  and  $z$  are equal when the plane bisects the second and fourth space quadrants, it follows that points belonging to the bisecting plane, when projected onto the planes  $\Pi_1$  and  $\Pi_2$ , merge in the multiview drawing as a single point.

### § 17. Level Lines and Lines of Maximum Inclination in a Plane

1. The horizontal of a plane. A straight line  $h$  contained in a given plane and parallel to the horizontal plane of projection  $\Pi_1$  is called a *horizontal of that plane*.

For instance, a horizontal cornice may serve as an example of the horizontal of the sloping plane of the roof.

A horizontal of a plane, as any horizontal in space, is projected onto the frontal plane of projection  $\Pi_2$  as a line parallel to the ground line  $Ox$ . Every point of a horizontal lies at the same altitude, for this reason they are also known as level lines.

In Fig. 103 two projections of triangle  $ABC$  are given. To construct the projections of a horizontal contained in the plane of this triangle, through the frontal projection of vertex  $C$  pass a frontal projection  $C_2D_2$  of the horizontal. This line is parallel to ground line  $Ox$ . Point  $D_1$  is found by passing a projector through point  $D_2$  as shown by an arrow. Then through  $C_1$  and  $D_1$  draw the horizontal

projection  $C_1D_1$  of the horizontal  $h$  contained in the plane triangle  $ABC$ .

If the plane is given by its traces (Fig. 104), the construction of a horizontal lying in it may also be begun by drawing the horizontal projection of the horizontal since the horizontal projection of the horizontal of a plane is parallel to the horizontal trace of that plane. Two straight lines, trace  $k$  and horizontal  $h$ , may be considered as the intersection of plane  $\alpha$  by two horizontal planes,  $\Pi_1$  and the plane at the level of the projecting horizontal itself.

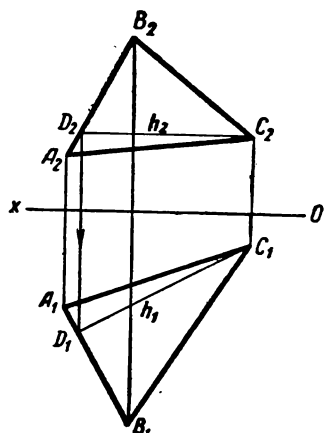


Fig. 103

When constructing the horizontal  $h$  in the multiview drawing (Fig. 104, b) it should be remembered that the traces of lines, lying in the plane, lie on the similar traces of the plane. Therefore first mark the projections ( $N_1$  and  $N_2 \equiv N$ ) of the frontal trace of the horizontal  $h$ . Then through point  $N_1$  draw the horizontal projection  $h_1$  of the horizontal. This line is parallel to the horizontal trace of plane  $\alpha$ . Next, through point  $N_2 \equiv N$  of the frontal projection, draw projection

$h_2$  of the horizontal parallel to ground line  $Ox$ .

2. The frontal of a plane. A straight line  $f$  contained in a given plane and parallel to the frontal plane of projection  $\Pi_2$  is called the *frontal of the plane*.

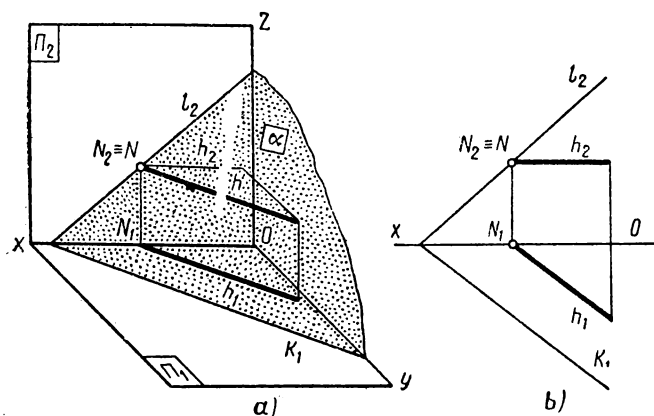


Fig. 104

From the drawing in Fig. 105 it may be seen that the horizontal projection  $f_1$  of the frontal is parallel to the ground line  $Ox$ . The frontal projection  $f_2$  of the frontal of the plane is parallel to the frontal trace  $l_2$  of plane  $\alpha$ , and the horizontal trace, point  $M$ , of the frontal of the plane lies on the horizontal trace of plane  $\alpha$ .

The construction of the projections of the frontal  $f$  in the plane

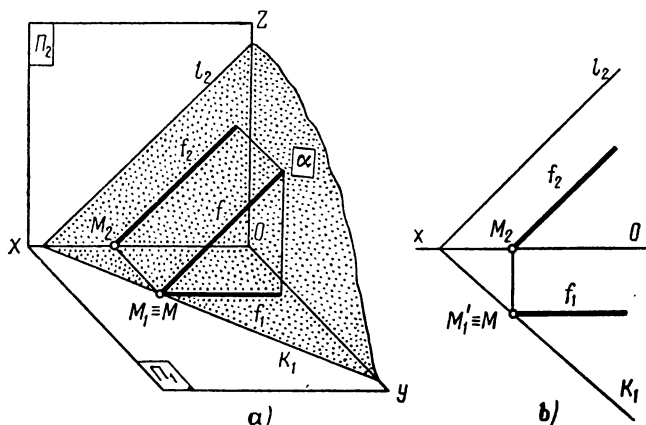


Fig. 105

given by the triangle  $ABC$ , is shown in Fig. 106. The construction is begun by drawing the horizontal projection  $f_1$  of the frontal through point  $B_1$ . This line is parallel to ground line  $Ox$ . The intersection of  $f_1$  with the extended projection  $A_1C_1$  of side  $AC$  of the triangle determines point  $D_1$ . Since point  $D_1$  has been found, the point  $D_2$  on the extension of projection  $A_2C_2$  can be determined. Join points  $B_2$  and  $D_2$ . The straight lines  $f_1$  and  $f_2$  thus obtained are the projections of frontal  $f$  of the plane of triangle  $ABC$ .

The projections of the frontals of the plane of triangle  $ABC$ , passing respectively through the vertex  $A$  and the vertex  $C$ , can likewise be constructed.

3. A *profile line* of a plane is a line contained in a given plane and parallel to the plane of projection  $\Pi_3$ .

In the multiview drawing of a plane, which is given by its traces, the construction of the projections of a profile line in the plane is best begun

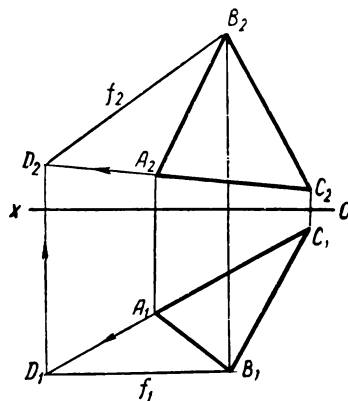


Fig. 106



by drawing its profile projection. This line is parallel to the profile trace of the plane. When a plane is given, say, by two parallel lines (Fig. 107), the construction should be begun by drawing the frontal projection of the profile line parallel to ground line  $Oz$ , or by drawing its horizontal projection parallel to ground line  $Oy_1$ .

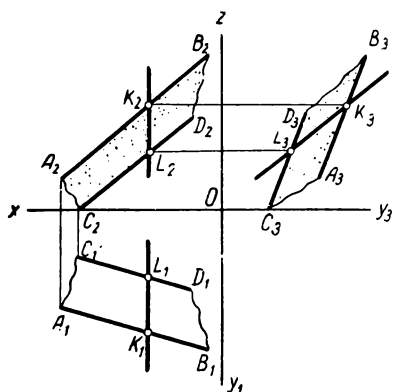


Fig. 107

Note. By analogy with the horizontals of a plane, the frontal and profile lines are also referred to as level lines.

4. In Figs. 108 to 111 the projections of horizontals and frontals contained in projecting planes are given. These are: the horizontal  $AB$  lying in a horizontal-projecting plane, the frontal  $CD$  in a horizontal-projecting plane, the horizontal  $KL$  in a frontal-projecting plane, and the

frontal  $EF$  lying in a frontal-projecting plane.

Example 1. Construct the horizontal projection of line  $AB$ , contained in plane  $\alpha$  and given by the frontal projection  $A_2B_2$  (Fig. 112).

Through points  $A_2$  and  $B_2$  we draw the frontal projections,  $A_2N_2$  and  $B_2N'_2$ , of the auxiliary horizontals of plane  $\alpha$ . Using points  $N_2$  and  $N'_2$ , determine the position on the ground line  $Ox$  of points  $N_1$  and  $N'_1$  and through them, parallel to the trace  $k_1$ , draw the horizontal projections  $A_1N_1$  and  $B_1N'_1$  of the auxiliary horizontals. By drawing projectors through points  $A_2$  and  $B_2$  points  $A_1$  and  $B_1$  are found. They lie at the intersections of the projectors with the horizontal projections of the corresponding auxiliary horizontals. The straight

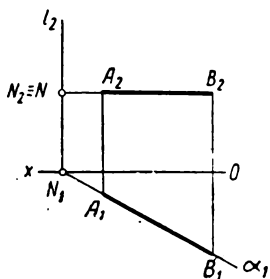


Fig. 108

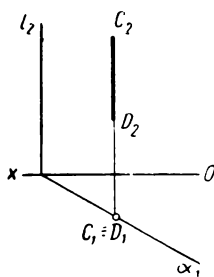


Fig. 109

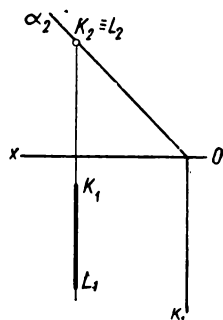


Fig. 110

line  $A_1B_1$  drawn through points  $A_1$  and  $B_1$  is then the horizontal projection of line  $AB$ .

**Example 2.** Construct a plane  $\alpha$  passing through point  $A$  and the point of convergence of traces  $X_\alpha$  (Fig. 113).

An infinite number of planes may be drawn through the given two points. The construction of one of those planes is as follows:

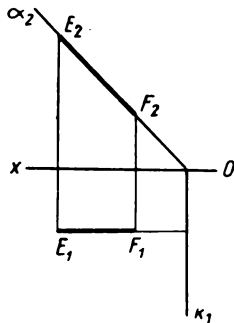


Fig. 111

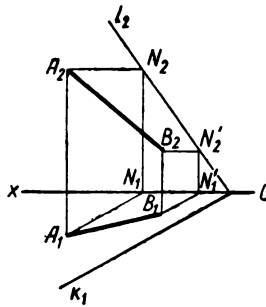


Fig. 112

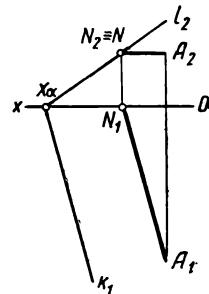


Fig. 113

Draw a horizontal through point  $A$ . Having found the trace  $N$  of this horizontal through point  $N_2 \equiv N$  and  $X_\alpha$  draw the frontal trace  $l_2$ . Draw the horizontal trace  $k_1$  of the required plane  $\alpha$  through the point  $X_\alpha$  parallel to  $A_1N_1$ .

**Example 3.** Construct the traces  $k_1$  and  $l_2$  of plane  $\alpha$  which is given by parallel lines  $AB$  and  $CD$  (Fig. 114).

First in the given plane draw the horizontal  $BD$  and frontal  $BC$ . Find trace  $N$  of the horizontal  $BD$ . Through point  $N_2$ , parallel to the projection of the frontal  $B_2C_2$ , draw the frontal trace  $l_2$  of plane  $\alpha$ .

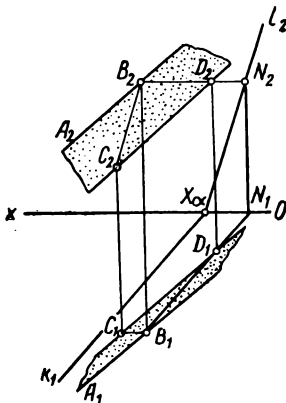


Fig. 114

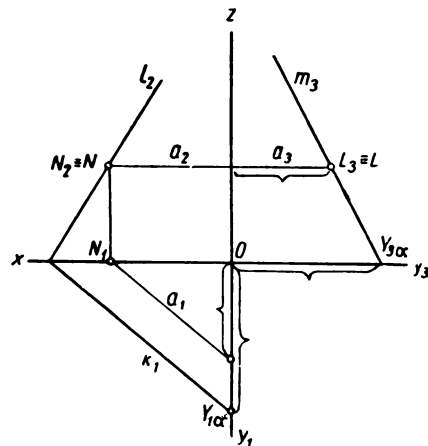


Fig. 115

and extend it to intersect the ground line  $Ox$  in point  $X_\alpha$ . Then, through point  $X_\alpha$  parallel to  $B_1D_1$  draw the horizontal trace  $k_1$  of the plane.

**Example 4.** Construct the profile trace  $m_3$  of plane  $\alpha$  which is given by traces  $k_1$  and  $l_2$ , the point of convergence  $Z_\alpha$  of traces being located beyond the limits of the drawing (Fig. 115).

Draw any horizontal in the plane  $\alpha$  and find its profile trace  $L$ . The point of convergence  $Y_{1\alpha}$  is carried over to ground line  $Oy_3$  as point  $Y_{3\alpha}$ . Join up the points  $L_3 \equiv L$  and  $Y_{3\alpha}$ . This line  $m_3$  is the profile trace of plane  $\alpha$ .

**5. Lines of maximum inclination of a plane.** A *line of maximum inclination of a plane* is a line contained in the given plane and perpendicular to its trace.

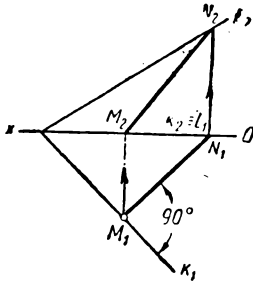


Fig. 116

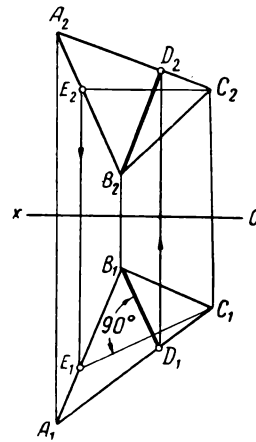


Fig. 117

**Hint.** The seam of metal sheet roofing perpendicular to the cornice may serve to illustrate what is meant by a line of maximum inclination of a plane.

In a given plane different lines of maximum inclination may be found, i. e., a) relative to the horizontal plane of projection  $\Pi_1$ , b) relative to the frontal plane of projection  $\Pi_2$ , and c) relative to the profile plane of projection  $\Pi_3$ .

In Fig. 116 the plane  $\alpha$  is shown together with the line of maximum inclination  $MN$  of the plane relative to the plane  $\Pi_1$ . The construction of the line  $MN \perp k$  is begun by drawing the horizontal projection  $M_1N_1 \perp k_1$  (see § 13). The frontal projection  $M_2N_2$  is drawn through points  $M_2$  and  $N_2$ , which are determined by points  $M_1$  and  $N_1$ . Point  $M_2$ , being the frontal projection of point  $M$  contained in the horizontal trace  $k_1$ , is located on the ground line  $Ox$ . The frontal projection  $N_2$  of point  $N$  lies at the point of intersection of trace  $l_2$  and the vertical projector drawn through point  $N_1$ .

The line of maximum inclination or slope of plane  $\alpha$  to plane  $\Pi_1$ , being perpendicular to the horizontal trace of plane  $\alpha$ , is also obviously perpendicular to any horizontal in the plane  $\alpha$ . Therefore the line of maximum inclination of plane  $\alpha$ , when the plane is given by triangle  $ABC$  (Fig. 117), is constructed with the help of an auxiliary horizontal drawn through the vertex  $C$  of the triangle. The horizontal projection  $B_1D_1$  of the line of maximum inclination is then passed through  $B_1$  perpendicular to the horizontal projection  $C_1E_1$  of the horizontal  $CE$ . The frontal projection is drawn through point  $B_2$  and point  $D_2$ , the latter being determined by a vertical projector passing through  $D_1$ .

The line of maximum inclination  $MN$  of plane  $\alpha$  relative to the frontal plane of projection  $\Pi_2$  is shown in Fig. 118. The frontal projection  $M_2N_2$  of this line  $MN$  is perpendicular to the trace  $l_2$ .

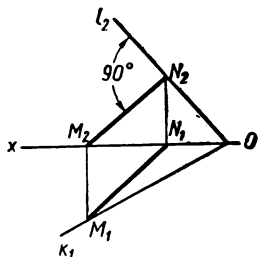


Fig. 118

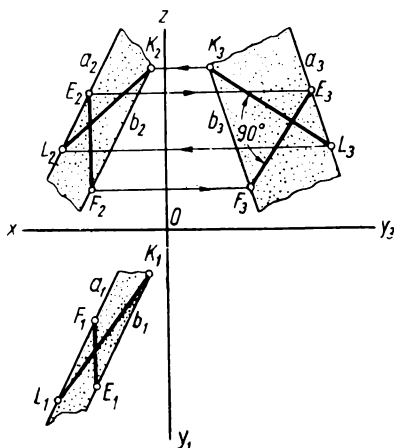


Fig. 119

The points  $M_1$  and  $N_1$  are constructed with the help of points  $M_2$  and  $N_2$ , as these are the projections of points lying on the traces  $k_1$  (point  $M$ ) and  $l_2$  (point  $N$ ).  $M_1N_1$  is the horizontal projection of the line of maximum inclination of plane  $\alpha$  to plane  $\Pi_2$ .

Line  $KL$  of maximum inclination of plane  $\gamma$  relative to plane  $\Pi_3$  (Fig. 119) is constructed in a similar manner. The profile projection  $K_3L_3$  will be perpendicular to the profile projection  $E_3F_3$  of the profile line  $EF$ . Then, by using points  $K_3$  and  $L_3$ , points  $K_2$  and  $L_2$ ,  $K_1$  and  $L_1$  are found. Through them draw the frontal projection  $K_2L_2$  and the horizontal projection  $K_1L_1$  of the line of maximum inclination of plane  $\gamma$  to plane  $\Pi_3$ .

6. With the assistance of lines of maximum inclination the angles of inclination of a given plane to the planes of projection may be determined.

These angles are given by the angles formed by the corresponding lines of maximum inclination to planes of projection  $\Pi_1$ ,  $\Pi_2$  and

$\Pi_3$ . The true value of these angles may be determined by the right triangle method (see § 8).

For example, determine the true magnitude of the angle of inclination of plane  $\alpha$  to the frontal plane of projection  $\Pi_2$  (Fig. 120).

In the plane  $\alpha$  construct a line  $MN$  perpendicular to the frontal trace  $l$ . This will be the line of maximum inclination of plane  $\alpha$  to

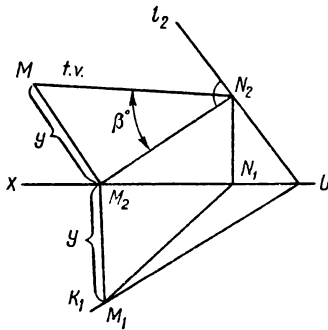


Fig. 120

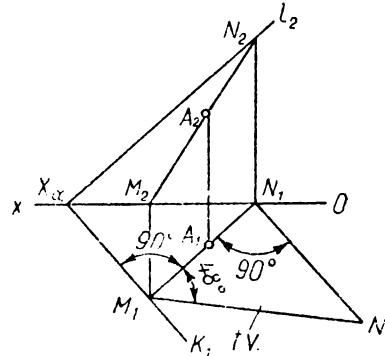


Fig. 121

plane  $\Pi_2$ . The frontal projection  $M_2N_2$  of this line is drawn perpendicular to trace  $l_2$ . Once the frontal projection is found, the horizontal projection  $M_1N_1$  is easily determined. It should be borne in mind that point  $N$  lies on the frontal trace and  $M$  on the horizontal trace. The projection  $M_1$  of the latter therefore lies on  $k_1$ .

The angle of inclination of line  $MN$  to plane  $\Pi_2$  is determined by constructing the right triangle  $M_2N_2\bar{M}$ . One side of this triangle is the projection  $M_2N_2$ . The second side is equal to the difference between the distances of points  $M$  and  $N$  from plane  $\Pi_2$ , i. e., to the value of  $y$  for point  $M_1$ . The angle  $\beta$  between the hypotenuse  $N_2\bar{M}$  of the resulting right triangle and projection  $M_2N_2$  is the angle of inclination of the line  $MN$  to plane  $\Pi_2$ . At the same time  $\beta^\circ$  is the true inclination of plane  $\alpha$  to that same plane  $\Pi_2$ .

7. By means of lines of maximum inclination a plane may be constructed so that it forms a given angle to the plane of projection.

The method of constructing a plane passing through a point  $A$  and inclined at  $48^\circ$  to the plane  $\Pi_1$  (Fig. 121) is given below.

First the line  $MN$ , passing through point  $A$  and making an angle of  $48^\circ$  to plane  $\Pi_1$ , should be constructed. In order to do this draw line  $M_1N_1$  at any angle to the ground line  $Ox$ . This line will serve as the side of the right triangle. Lay off from this side two angles — an angle of  $48^\circ$  from point  $M_1$  and an angle of  $90^\circ$  from point  $N_1$ .

A distance equal to the length of the side  $N_1\bar{N}$  is then laid off from point  $N_1$  on the perpendicular above ground line  $Ox$ . The point  $N_2$  thus found is connected to point  $M_2$  which is obtained by drawing through point  $M_1$  a perpendicular to ground line  $Ox$  to their intersection.

The lines  $M_1N_1$  and  $M_2N_2$  are the projections on  $\Pi_1$  and  $\Pi_2$  of line  $MN$  drawn through point  $A$  at an angle of  $48^\circ$  to plane  $\Pi_1$ .

Next, through this line  $MN$  a plane  $\alpha$  should be drawn so that  $MN$  serves as its line of maximum inclination relative to  $\Pi_1$ . In order to do this draw the trace  $k_1$  at an angle  $90^\circ$  to the projection  $M_1N_1$  and the trace  $l_2$  passing through the point  $X_\alpha$  of convergence of the traces and point  $N_2$ .

### Problems and Exercises

1. How may a plane be determined?
2. What is a trace of a plane?
3. What is meant by the term an "oblique plane"?
4. What is meant by the term a "projecting plane"?

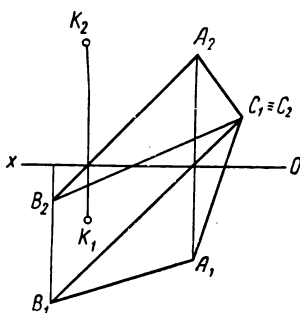


Fig. 122

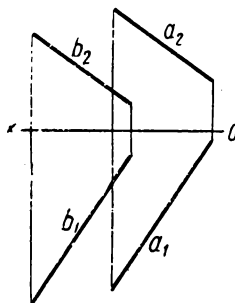


Fig. 123

5. Describe the positions of the traces of a plane lying parallel to the ground line  $Ox$ .
6. When do the traces of planes on the multiview drawing merge into one line cutting through ground line  $Ox$ ?
7. How are the traces of a plane constructed when it is given by two intersecting lines?
8. How is a point lying in an oblique plane constructed?
9. How can it be checked whether a point is contained in a plane?
10. What is meant by a "horizontal of a plane"?
11. How is a plane constructed to pass through a given point?
12. What is meant by a "line of maximum inclination of a plane"?

13. How is the angle of inclination of a plane to the horizontal plane of projection determined?

14. Is the point  $K$  contained in the plane of the triangle  $ABC$  (Fig. 122)?

Hint. Through projections  $A_1$  and  $K_1$  draw the projection of an auxiliary straight line contained in the plane of the triangle  $ABC$ .

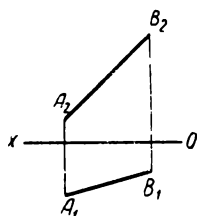


Fig. 124

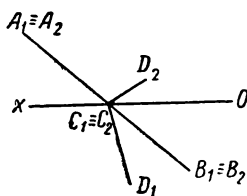


Fig. 125

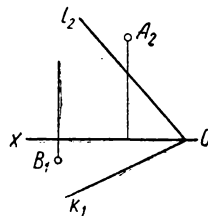


Fig. 126

When the frontal projection of this line is found, it may be seen that the line constructed in the plane  $ABC$  does not pass through point  $K$  in space.

15. Construct the traces of a plane given by two parallel lines (Fig. 123).

Hint. It is convenient to draw a horizontal and a frontal in the given plane and find the trace of one of them. This is of course not the only method. Any line lying in the given plane may be used, including lines  $a$  and  $b$ , but the method recommended is the most simple.

16. Through a given point  $A$  draw a plane  $\alpha$ , the traces of which merge into one straight line.

17. Through line  $AB$  draw an oblique plane and a horizontal-projecting plane (Fig. 124).

18. Construct the traces of the plane given by the intersecting lines  $AB$  and  $CD$  (Fig. 125). (See exercise 15.)

19. Construct the projection of triangle  $ABC$  contained in the given plane (Fig. 126). Use lines of altitude.

20. Plane  $\alpha$  is given by point  $A$  and a frontal trace. Construct the horizontal trace of the plane (Fig. 127).

Hint. Through point  $A$  in the given plane draw a horizontal.

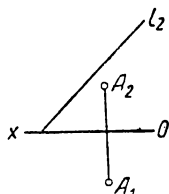


Fig. 127

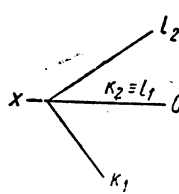


Fig. 128

Remember that the traces of lines lie on the corresponding traces of its planes.

21. Determine the angles of inclination of the plane of triangle  $ABC$  to the planes of projection  $\Pi_2$  and  $\Pi_3$  (Fig. 117).

Hint. To solve the problem it is necessary to draw in the plane of triangle  $ABC$  the frontal and profile lines and, perpendicular to them, lines of maximum inclination of plane  $ABC$  relative to the planes of projection  $\Pi_2$  and  $\Pi_3$  (see text pertaining to Fig. 117).

22. Determine the angles of inclination of plane  $\alpha$  to planes of projection  $\Pi_1$  and  $\Pi_3$  (Fig. 128).

23. A plane  $\alpha$  passing through ground line  $Ox$  and through a point  $A$  is given, as is the frontal projection  $B_2C_2$  of line  $BC$  contained in this plane. Construct the horizontal projection  $B_1C_1$  without using the profile plane of projection  $\Pi_3$ .

Hint. To solve this problem it is necessary to draw an auxiliary line intersecting  $BC$  through point  $A$  of plane  $\alpha$ .

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## Chapter V

### LINES AND PLANES

#### § 18. Straight Lines Parallel to Planes. Parallel Planes

1. In elementary geometry it is demonstrated that a straight line is parallel to a given plane if the line is parallel to at least one line contained in the plane, or if it lies in a plane parallel to the given plane.

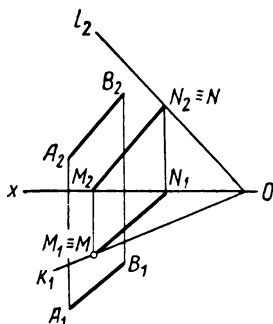


Fig. 129

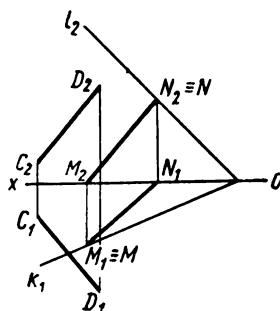


Fig. 130

The corresponding projections of two parallel lines are also parallel. It follows that the line  $AB$  in Fig. 129 is parallel to the plane  $\alpha$ , since the projections  $A_1B_1$  and  $A_2B_2$  of this line are parallel to the corresponding projections  $M_1N_1$  and  $M_2N_2$  of line  $MN$  contained in plane  $\alpha$ .

The line  $CD$ , shown in Fig. 130, is not parallel to plane  $\beta$  because it is not possible to draw in the given plane  $\beta$  a line parallel to the given line  $CD$ . This can be demonstrated as follows: parallel to projections  $C_2D_2$  draw the projection  $M_2N_2$  of line  $MN$  contained in plane  $\beta$  and construct the projection  $M_1N_1$  from projection  $M_2N_2$ . It is seen that  $M_1N_1$  is not parallel to the projection  $C_1D_1$ . This means that line  $CD$  is not parallel to the plane  $\beta$ .

2. If it is required to draw through point  $K$  (Fig. 131) a straight line  $AB$  parallel to the plane of triangle  $CDE$ , first in the plane of the triangle draw an auxiliary line  $EF$ . Then through point  $K$  draw a line  $AB$  parallel to  $EF$ , i. e., so that  $A_1B_1 \parallel E_1F_1$  and  $A_2B_2 \parallel E_2F_2$ . The constructed line  $AB$  is parallel to the plane of the triangle  $CDE$ . Another solution would be to draw through point  $K$  a line

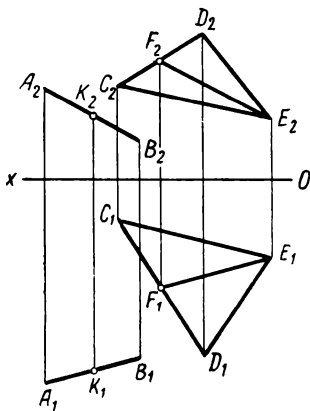


Fig. 131

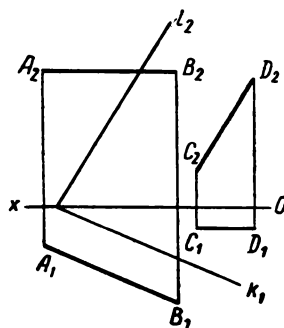


Fig. 132

parallel to one of the sides of the triangle. In this case it is not necessary to construct an auxiliary line in the plane of the triangle.

The construction of horizontal and frontal lines parallel to a given plane, when it is defined by its traces, is also carried out directly, without using auxiliary lines, since the lines required will be parallel to the respective traces. The construction of the horizontal  $AB$  parallel to the horizontal trace  $k$  and frontal  $CD$  parallel to the frontal trace  $l$  of plane  $\alpha$  is shown in Fig. 132. Lines  $AB$  and  $CD$  are parallel to  $\alpha$ .

3. The construction of a plane parallel to a given line is carried out with the help of an auxiliary line. A plane passing through an auxiliary line drawn parallel to the given line will be parallel to the latter.

In order to construct a plane  $\alpha$  parallel to the given line  $AB$  and passing through point  $C$  (Fig. 133) draw a straight line  $CD$  of any suitable length parallel to  $AB$  and join the points  $C$  and  $D$  with any point  $E$ . The plane defined by triangle  $CDE$  is one of the possible planes parallel to line  $AB$  passing through point  $C$ .

To construct the traces of a plane which is parallel to a given line  $AB$  and passes through a given point  $K$  (Fig. 134), through point  $K$  draw an auxiliary line  $KL$  parallel to  $AB$ . Then construct the traces  $M$  and  $N$  of the line  $KL$ . Through these traces and

a point taken at random on the ground line  $Ox$ , which is assumed as  $X_\alpha$ , the point of convergence of the traces, draw the traces  $k_1$  and  $l_2$  of the required plane  $\alpha$ , which is one of the planes satisfying the given conditions. If the traces of the plane were to be drawn through points  $M$  and  $N$  parallel to ground line  $Ox$ , then the plane constructed would be parallel to line  $KL$  and to ground line  $Ox$ .

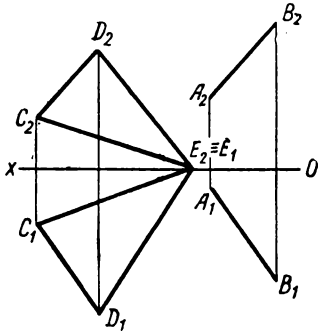


Fig. 133

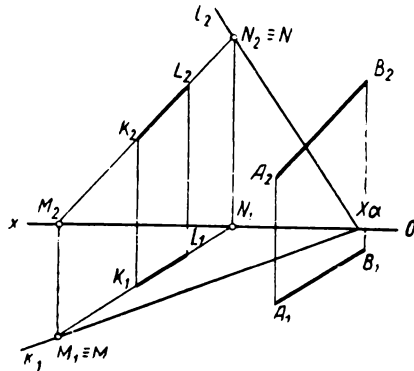


Fig. 134

4. The lines of intersection of two mutually parallel planes with any third plane are parallel lines (Fig. 135, where  $AB \parallel CD$ ). Since the traces of planes are straight lines along which the given planes intersect the planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$ , corresponding traces of parallel planes are parallel to each other (Fig. 136, where  $k_1 \parallel k'_1$  and  $l_2 \parallel l'_2$ ).

When the traces of two planes are not given, these planes are recognized as parallel, if two intersecting lines contained in the first plane are respectively parallel to two intersecting lines contained in the second plane. Horizontal and frontal lines may serve as the two intersecting lines in each of the planes. The two planes given by triangles  $ABC$  and  $DEF$  (Fig. 137) are parallel to each other since the corresponding projections of their horizontals

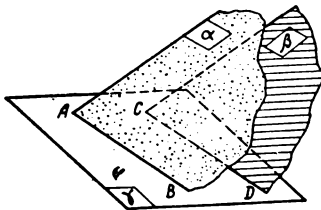


Fig. 135

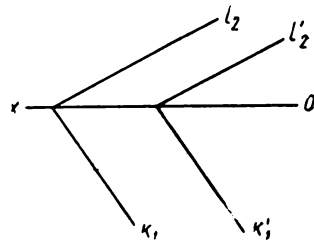


Fig. 136

$CM$  and  $DK$  and of the frontals  $AN$  and  $EL$  are parallel ( $CM \parallel D_1K_1$  and  $A_2N_2 \parallel E_2L_2$ ).

5. Even if two planes are parallel to ground line  $Ox$  and their horizontal and frontal traces are parallel to that ground line, it may well happen that such planes are not parallel to each other. To determine whether such planes are really parallel it is necessary to construct their profile traces or to check whether the ratios of the distances of corresponding traces from the ground line  $Ox$  are equal.

The planes  $\alpha$  and  $\beta$ , shown in Fig. 138, are not parallel to each other, although their corresponding traces in  $\Pi_1$  and  $\Pi_2$  are parallel ( $k_1 \parallel k'_1$  and  $l_2 \parallel l'_2$ ). As can be seen the profile traces  $\alpha_3$  and  $\beta_3$  intersect.

The planes  $\alpha$  and  $\beta$ , shown in Fig. 139, are parallel to each other, since the corresponding pairs of traces on the planes of projection  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  are parallel. Due to this the ratios mentioned above are equal, e. g.,  $m : n = q : s$ .

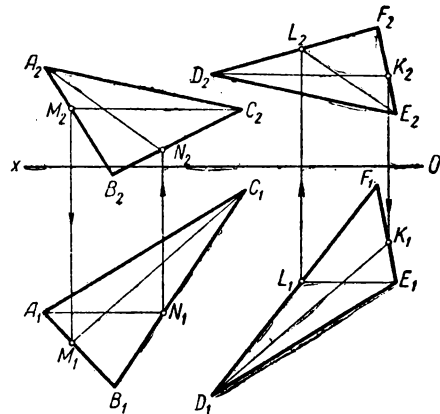


Fig. 137

6. The method of drawing a plane parallel to a given plane is based on the fact that the plane to be construct-

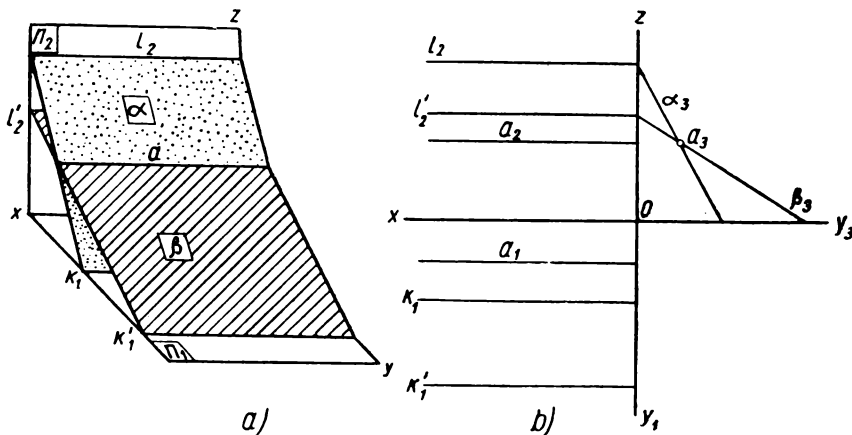


Fig. 138

ed should contain two intersecting lines respectively parallel to two intersecting lines contained in the given plane.

Let plane  $\alpha$  be given by a triangle  $ABC$ . It is required to draw a plane  $\beta$  parallel to plane  $\alpha$  and passing through point  $K$  (Fig. 140). The construction of the required plane may be carried out with the help of two intersecting lines parallel to two

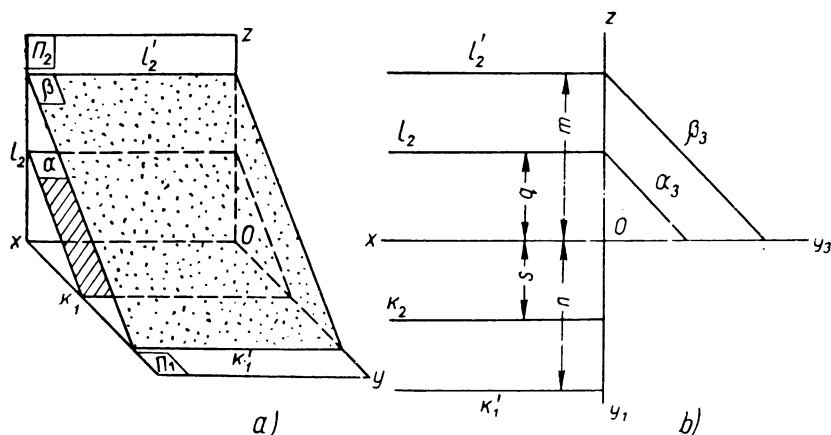


Fig. 139

corresponding lines contained in the plane of triangle  $ABC$ . Two sides of the triangle may serve as such lines. Draw through point  $K$  line  $a \parallel AC$  and line  $b \parallel BC$ . The plane  $\beta$ , passing through intersecting lines  $a$  and  $b$ , will be the required plane.

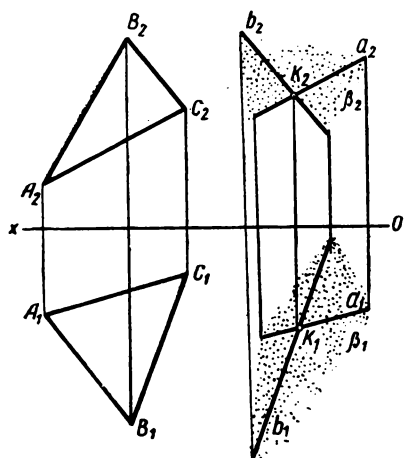


Fig. 140

When plane  $\alpha$  is given by its traces  $k_1$  and  $l_2$  and it is required to draw through point  $A$  plane  $\beta$  parallel to plane  $\alpha$  the construction is carried out as follows (Fig. 141):

1. Through point  $A$  draw a horizontal line  $AN$  parallel to plane  $\alpha$  ( $A_1N_1 \parallel k_1$  and  $A_2N_2 \parallel Ox$ ).

2. Find the frontal trace  $N$  of this line.

3. Through point  $N_2 \equiv N$  draw the trace  $n_2$  parallel to  $l_2$ .

4. Through point  $X_\beta$  draw the trace  $m_1$  parallel to trace  $k_1$ . The plane  $\beta$  is then parallel to the given plane  $\alpha$ .

Instead of using a line

parallel to the plane of projection when solving this problem (Fig. 141) it would, of course, also be possible to use oblique lines.

7. A plane parallel to a given plane can be made to pass through a given line only if the given plane and line are mutually parallel.

The plane  $\alpha$  and the line  $a$  parallel to plane  $\alpha$  ( $a \parallel \alpha$ ) are given. It is required to draw through line  $a$  a plane  $\beta$  parallel

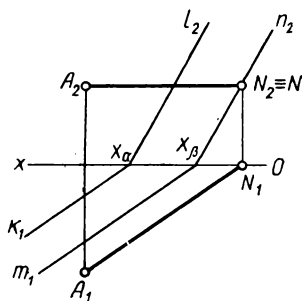


Fig. 141

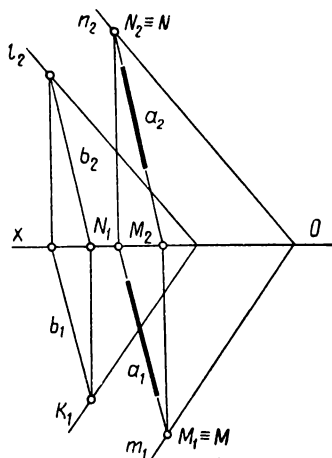


Fig. 142

to plane  $\alpha$  (Fig. 142). Construct the traces  $M$  and  $N$  of line  $a$  and through them draw the corresponding traces  $m_1$  and  $n_2$  of plane  $\beta$  parallel to the traces  $k_1$  and  $l_2$  respectively. The plane  $\beta$  constructed in this way is the required plane.

### Problems and Exercises

1. How is it possible to determine whether a given line and plane are parallel?
2. How may a plane be drawn parallel to a given plane?
3. How may a plane be drawn parallel to a given line?
4. Is the line  $a$  parallel to plane  $\alpha$  (Fig. 143)?
5. Through point  $A$  draw a line parallel to the horizontal-projecting plane (Fig. 144).
6. Through point  $A$  draw a plane parallel to plane  $\alpha$  which is parallel to the ground line  $Ox$  (Fig. 145).

Hint. First, in the given plane draw an auxiliary straight line passing through two points taken on the traces  $k$  and  $l$ . Then, having drawn through point  $A$  a second line, parallel to the auxiliary line, find the traces of this second line through which the traces of the required plane will pass (see Fig. 142 and text).

7. Construct the horizontal projection of triangle  $ABC$  the plane of which is parallel to the horizontal-projecting plane  $\alpha$  (Fig. 146).

8. Through point  $D$  draw a plane  $\alpha$  parallel to the plane given by the intersecting lines  $AB$  and  $BC$  (Fig. 147). The plane should be represented as a triangle.

9. Construct the traces of three mutually parallel planes  $\alpha$ ,  $\beta$ ,

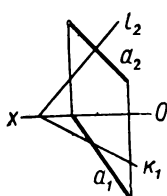


Fig. 143

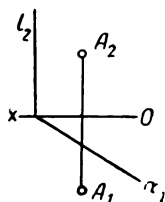


Fig. 144

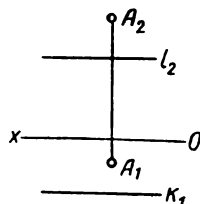


Fig. 145

and  $\gamma$  passing respectively through the given points  $A$ ,  $B$  and  $C$  (Fig. 148).

*Hint.* Three mutually parallel horizontal lines, drawn through points  $A$ ,  $B$  and  $C$  and their traces, should be used.

10. Through a given point  $A$  draw a plane parallel to a plane passing through ground line  $Ox$  and a point  $B$ . (See exercise 6).

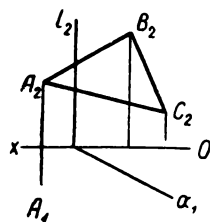


Fig. 146

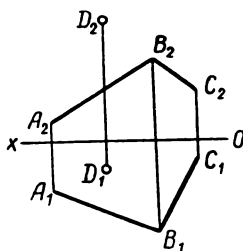


Fig. 147

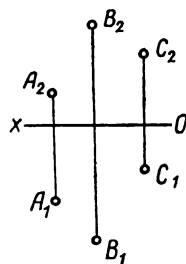


Fig. 148

## § 19. The Intersection of Planes

1. The intersection of two planes is a straight line defined by two points which are common to both planes. So the construction of the projections of the line of intersection of two planes, as a rule, can be reduced to the determination of the projections of two points, lying in both intersecting planes.

The method of determining points which lie on the line of intersection is called the *method of auxiliary sections*. It is widely used in all courses of descriptive geometry.

The application of the method of auxiliary sections is illustrated in Fig. 149, which shows the plane  $\alpha$ , given by the parallel lines

$AB$  and  $CD$  and the plane  $\beta$ , given by the triangle  $EFM$ . Points  $K$  and  $L$ , common to both planes  $\alpha$  and  $\beta$ , are determined by using two auxiliary planes,  $\gamma$  and  $\varphi$ . The plane  $\gamma$  intersects planes  $\alpha$  and  $\beta$  in lines  $1-2$  and  $3-4$ , respectively, and plane  $\varphi$  in lines  $5-6$  and  $7-8$ . The lines  $1-2$  and  $3-4$  intersect in point  $K$  common to planes  $\alpha$ ,  $\beta$  and  $\gamma$ . Therefore point  $K$  lies on the line of intersection of planes  $\alpha$  and  $\beta$ . Point  $L$  is common to planes  $\alpha$ ,  $\beta$  and  $\gamma$  and lies on the line of intersection of planes  $\alpha$  and  $\beta$  which is found by connecting up points  $K$  and  $L$ .

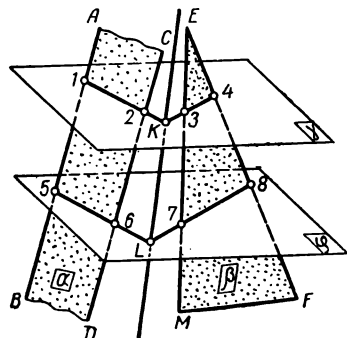


Fig. 149

This method of auxiliary sections is used for solving problems in the multiview drawing. It is required to find the line of intersection of two planes,  $\alpha$  and  $\beta$ , given by the pairs of parallel lines  $AB$  and  $CD$  and  $EF$  and  $MN$ , respectively (Fig. 150).

Begin the construction by drawing an auxiliary horizontal plane  $\gamma$  to intersect both planes.

The plane  $\gamma$  intersects plane  $\alpha$  in the horizontal  $1-2$ , the frontal projection  $1_2-2_2$  of which coincides with trace  $\gamma_2$ . The horizontal projection  $1_1-2_1$  is easily found with the help of projectors  $1_2-1_1$  and  $2_2-2_1$ . The projections of the horizontal  $3-4$ , along which the plane  $\beta$  is intersected by plane  $\gamma$ , are found in a similar manner. Point  $K_1$  lies at the intersection of the horizontal projections  $1_1-2_1$  and  $3_1-4_1$  of the horizontals. Point  $K_1$  is the horizontal projection of point  $K$  which belongs to the line of intersection of the given planes  $\alpha$  and  $\beta$ . The frontal projection  $K_2$  of point  $K$  is obtained by drawing

through  $K_1$  a projector to intersect the trace  $\gamma_2$ . This trace possesses collecting properties.

To construct the second point on the line of intersection of planes  $\alpha$  and  $\beta$  a second auxiliary plane  $\varphi$  is drawn. Its intersections with the given planes  $\alpha$  and  $\beta$  are also horizontals. Since all the horizontals of one plane are parallel to each other, in order to construct the horizontals in which plane  $\varphi$  intersects planes  $\alpha$  and  $\beta$  it is

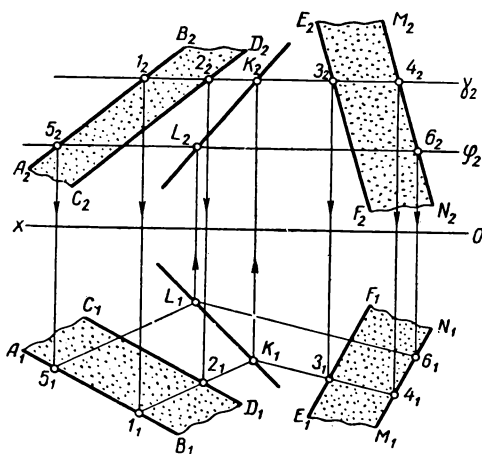


Fig. 150



sufficient to find one point of each of these two horizontals. Having constructed points  $5_1$  and  $6_1$  in  $\Pi_1$  draw through them the horizontal projections of the horizontals. They are respectively parallel to the projections  $1_1 2_1$  and  $3_1 4_1$  of the horizontals previously constructed. Point  $L_1$  lies at the intersection of the lines drawn through  $5_1$  and  $6_1$ . This is the horizontal projection of point  $L$

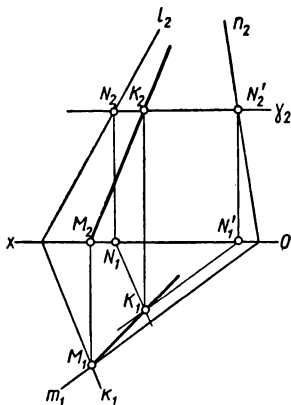


Fig. 151

through which the line of intersection of planes  $\alpha$  and  $\beta$  passes. The frontal projection  $L_2$  of point  $L$  is found with the help of a projector drawn through point  $L_1$  to the trace  $\varphi_2$  of the second auxiliary plane. By joining up the corresponding projections of points  $K$  and  $L$  two projections of  $KL$ , the line of intersection of the given planes  $\alpha$  and  $\beta$ , are found.

It should be noted that, instead of auxiliary horizontal planes, any two projecting planes could be used without changing the sequence of the construction.

2. When the position of one of the points contained in the line of intersection of two planes is known the construction is simplified.

In Fig. 151 planes  $\alpha$  and  $\beta$  are given. Their horizontal traces  $k_1$  and  $m_1$  intersect within the limits of the drawing. The point  $M$ , common to the planes  $\alpha$  and  $\beta$ , is one of the points belonging to the line of intersection of these planes. The horizontal projection of this point coincides with the point itself ( $M_1 \equiv M$ ). The frontal projection  $M_2$  lies on ground line  $Ox$ .

To find the second point, common to both planes  $\alpha$  and  $\beta$ , draw an auxiliary horizontal plane  $\gamma$  the intersection of which with each of the given planes will be a horizontal. The frontal projection of these horizontals will coincide with the frontal trace  $\gamma_2$  of plane  $\gamma$ , while the horizontal projections will be parallel to the horizontal traces  $k_1$  and  $m_1$  of planes  $\alpha$  and  $\beta$ , respectively, and pass through the horizontal projections  $N_1$  and  $N'_1$  of their frontal traces  $N$  and  $N'$ .

The horizontal projection  $K_1$  of point  $K$  contained in the line of intersection of planes  $\alpha$  and  $\beta$  lies at the intersection of horizontal projections  $N_1 K_1$  and  $N'_1 K_1$  of the horizontals constructed. The frontal projection  $K_2$  of point  $K$  is found on  $\gamma_2$  by drawing a projector through  $K_1$ . The projections of the line of intersection  $MK$  of the planes  $\alpha$  and  $\beta$  are found by joining the projections of point  $M$  with the corresponding projections of point  $K$ .

It should be noted that the use of an auxiliary horizontal plane

$\gamma$  is convenient because the construction of the horizontals in planes  $\alpha$  and  $\beta$  requires only one piercing point to be determined for each of those lines, since the horizontal projection of the horizontal of a plane is parallel to the horizontal trace of that plane — see § 17 and Fig. 104.

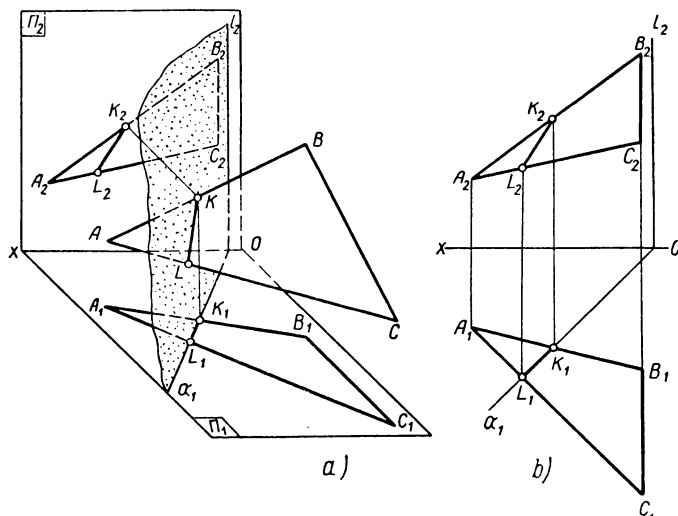


Fig. 152

3. The construction of the line of intersection of two planes is further simplified if at least one of the planes is a projecting plane. Such a case is shown in Fig. 152, where an oblique plane given by triangle  $ABC$  intersects the horizontal-projecting plane  $\alpha$ . Since the horizontal trace of the horizontal-projecting plane possesses collecting properties, the horizontal projection  $K_1L_1$  of the intersection line  $KL$  for the given planes coincides with trace  $\alpha_1$  of plane  $\alpha$ . Mark on this trace-projection the points  $K_1$  and  $L_1$  in which the horizontal projections  $A_1B_1$  and  $A_1C_1$  of the sides of the triangle intersect with  $\alpha_1$ . Draw through these points projectors to intersect the frontal projections  $A_2B_2$  and  $A_2C_2$  of the sides of the triangle in points  $K_2$  and  $L_2$ . The straight line joining these points is the frontal projection  $K_2L_2$  of the line of intersection of the two given planes.

4. The intersection of planes  $\alpha$  and  $\beta$ , given by their traces is shown in Fig. 153. The corresponding traces of these planes intersect within the limits of the drawing. The construction of the line of intersection  $MN$  consists merely in determining the second projections of points  $M$  and  $N$  in which the similar traces of the

planes intersect. The frontal projection  $M_2$  of point  $M$  and the horizontal projection  $N_1$  of point  $N$  both lie on the ground line  $Ox$ , as they are projections of points situated in the planes of projections. The projections of the line  $MN$ , i. e., the line of intersection of planes  $\alpha$  and  $\beta$ , are obtained by joining up the corresponding projections of points  $M$  and  $N$ .

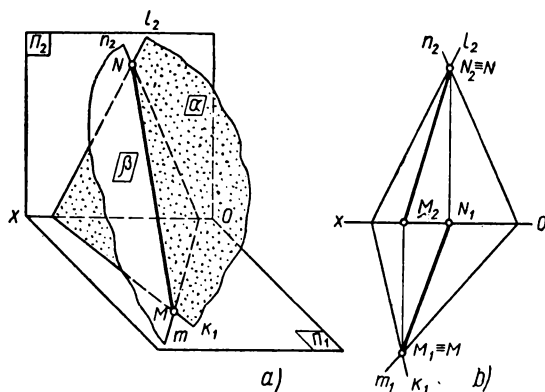


Fig. 153

5. The construction of the line of intersection of the frontal plane with the oblique plane  $\alpha$  (Fig. 154) consists in determining the frontal line common to both given planes. The point  $M$  where the horizontal traces  $k_1$  and  $\beta_1$  intersect is the horizontal trace of this frontal. Projection  $M_1$  coincides with the point itself, whereas the frontal projection  $M_2$  of point  $M$  lies on the ground line  $Ox$ . The line  $M_2L_2$  parallel to trace  $l_2$  of plane  $\alpha$  is the frontal projection of line  $ML$  of intersection of planes  $\alpha$  and  $\beta$ . The horizontal

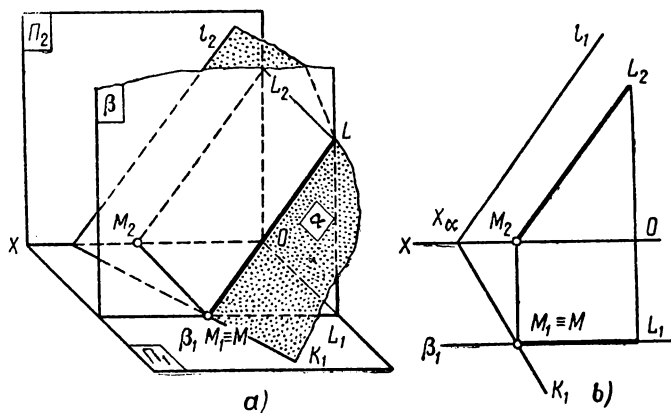


Fig. 154

projection  $M_1 L_1$  of the line of intersection of these planes coincides with trace  $\beta_1$ .

6. The cases of intersection of two planes considered above may be divided into two groups.

The first group consists of pairs of planes, the construction of the line of intersection of which requires the construction of two points contained in that line (Figs. 150, 152). The second group consists of pairs of planes whose lines of intersection are determined by one point and a known direction (Fig. 154).

It should be noted that when determining the direction of the lines of intersection of two planes an auxiliary plane parallel to one of those planes may be used.

For example, it is required to construct the line of intersection of two oblique planes  $\alpha$  and  $\beta$  which are given by their traces. The horizontal traces  $k_1$  and  $m_1$  of these planes intersect within the limits of the drawing (Fig. 155).

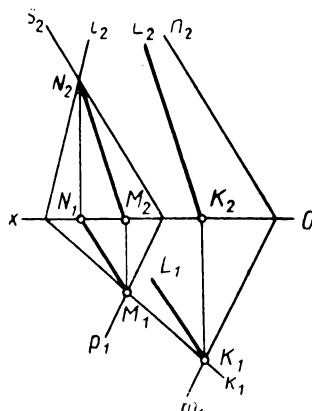


Fig. 155

Draw an auxiliary plane  $\gamma$  parallel to plane  $\beta$  so that its traces  $p_1$  and  $s_2$ , which will be parallel to traces  $m_1$  and  $n_2$ , respectively, intersect the corresponding traces of plane  $\alpha$ . Having obtained points  $M$  and  $N$ , where the traces of planes  $\alpha$  and  $\gamma$  intersect, construct the projections  $M_1N_1$  and  $M_2N_2$  of the line of intersection  $MN$  of these planes. Since the intersection lines of two parallel planes with a third plane are themselves two parallel lines, the line  $MN$  determines the direction to which the line of intersection of the given planes  $\alpha$  and  $\beta$  will be parallel. Having found point  $K$ , common to both of these planes, i. e., the point in which their horizontal traces intersect, through it draw line  $KL \parallel MN$ . This line  $KL$  is the required line of intersection of planes  $\alpha$  and  $\beta$ .

7. While working on problems of graphical construction the drawing should be kept as free as possible from auxiliary lines. The following example shows how to achieve this.

Plane  $\alpha$  is given by its traces  $k_1$  and  $l_2$  and plane  $\beta$  by the triangle  $ABC$ . It is required to construct the projections of the line of intersection of these planes (Fig. 156).

Two auxiliary frontal planes  $\gamma$  and  $\varphi$  will be used. Each of these auxiliary planes intersects the given planes  $\alpha$  and  $\beta$  in frontals. The point of their intersection will lie on line of intersection of planes  $\alpha$  and  $\beta$ .

Let the first auxiliary plane  $\gamma$  pass through the vertex  $C$  of the triangle. This plane will cut plane  $\alpha$  along the frontal  $MS$  and the plane of triangle  $ABC$  along the frontal  $CD$ . The intersection of the frontals  $MS$  and  $CD$  will determine the position of the first point  $K$  through which the required line of intersection of the given planes will pass.

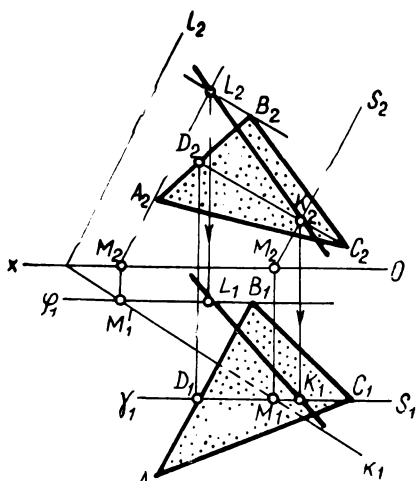


Fig. 156

The second auxiliary plane  $\varphi$  is drawn through point  $B$  parallel to the frontal plane of projection  $\Pi_2$ . The frontal projection, which is the line of intersection of planes  $\varphi$  and  $\alpha$ , will pass through point  $M'_2$  and be parallel to trace  $l_2$  of plane  $\alpha$ . The frontal projection of the frontal, which is the line of intersection of the plane  $\varphi$  and the triangle  $ABC$ , will pass through point  $B_2$  and be parallel to the projections  $C_2D_2$  of the frontal  $CD$  previously constructed. The intersection of the frontals in which the plane  $\varphi$  intersects the planes  $\alpha$  and  $\beta$

will define point  $L$ , lying on the line of intersection of planes  $\alpha$  and  $\beta$ . The line  $KL$  is the required line of intersection of the two given planes.

It should be noted that, in order to simplify the construction in the drawing, plane  $\gamma$  was passed through one of the vertices of triangle  $ABC$ . So when constructing the frontal projection  $C_2D_2$  of the frontal  $CD$  only one point  $D_2$  has to be found. The second auxiliary plane  $\varphi$  is passed through the vertex  $B$  of triangle  $ABC$ ,

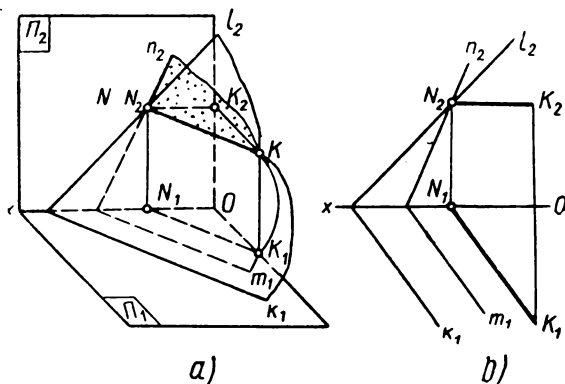


Fig. 157

again simplifying the construction, since the frontal projection of the second frontal of the plane of the triangle was then drawn through point  $B_2$  parallel to line  $C_2D_2$ .

8. Finally attention should be paid to a particular case of intersection of two oblique planes when two of the corresponding traces of the planes are parallel.

In Fig. 157 the line of intersection  $NK$  of planes  $\alpha$  and  $\beta$  is parallel to the horizontal traces of the given planes since these traces are parallel. In other words, the line of intersection is the horizontal common to both planes. Similarly, if the frontal traces of two intersecting planes are parallel, the line of intersection is the frontal common to both of the planes and, if the profile traces of the planes are parallel, the line of intersection is the profile line common to the planes.

#### Problems and Exercises

1. Why are auxiliary planes used when constructing the line of intersection of two planes?

2. Which planes are most frequently used as auxiliary planes?

3. Give an example of the use of an oblique plane as an auxiliary plane.

4. Where do the traces of a line lie when two planes intersect on that line?

5. Construct the line of intersection of two planes, one of which is given by the triangle  $ABC$  and the second by two parallel lines  $a$  and  $b$  (Fig. 158).

Hint. Use two auxiliary horizontal or frontal section planes. It is not advisable to use one horizontal plane and one frontal plane, since the solution of the problem would require an unnecessary number of construction lines which would overload the drawing.

6. Construct the lines of intersection of the frontal-projecting plane  $\alpha$  with the oblique plane  $\beta$  (Fig. 159).

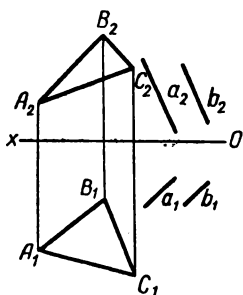


Fig. 158

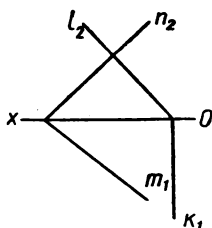


Fig. 159

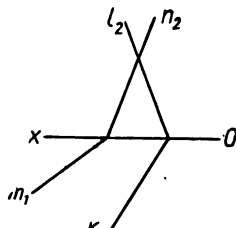


Fig. 160

7. Construct the line of intersection of two oblique planes (Fig. 160).

*Hint.* Use an auxiliary frontal section plane.

8. Find the horizontal traces of planes  $\alpha$  and  $\beta$  whose line of intersection passes through point  $M$  (Fig. 161).

*Hint.* The line of intersection of planes, when their two corresponding traces are parallel to each other, is also parallel to the traces. In this case the line of intersection is the frontal passing through point  $A$  parallel to traces  $l$  and  $n$ . The required horizontal traces of the planes  $\alpha$  and  $\beta$  will pass through the trace of the given frontal.

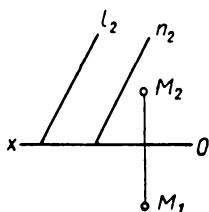


Fig. 161

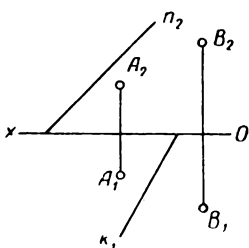


Fig. 162

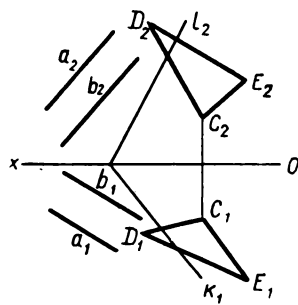


Fig. 163

9. Construct the line of intersection of planes  $\alpha$  and  $\beta$ . The plane  $\alpha$  is given by the horizontal trace  $k_1$  and point  $A$ . Plane  $\beta$  is given by the frontal trace  $n_2$  and point  $B$  (Fig. 162).

*Hint.* First find the traces of the given planes. The trace of plane  $\alpha$  is found with the help of the horizontal of the plane passing through point  $A$ . The trace of plane  $\beta$  is found with the help of the frontal passing through point  $B$ .

10. Find the point common to three planes, one of which is given by triangle  $CDE$ , the second, by parallel lines  $a$  and  $b$ , and the third by traces  $k_1$  and  $l_2$  (Fig. 163).

*Hint.* This problem can be reduced to the construction of two lines of intersection of two pairs of planes. For example, the line of intersection of plane  $CDE$  and the plane given by traces  $k$  and  $l$  can be found. Then the line of intersection of the plane given by the traces  $k$  and  $l$  and the plane given by two parallel lines can be determined. The construction should be worked out with the help of two auxiliary horizontal planes, the first of which should pass through point  $E$  and the second, through point  $C$ .

11. Construct the line of intersection of two oblique planes the points of convergence of the frontal and horizontal traces of which converge in one point.

Hint. One point belonging to the line of intersection of the planes is the point of convergence of the four traces. Another point is found with the help of horizontal and frontal auxiliary sections.

## § 20. The Intersection of Lines and Planes

1. Plane  $\alpha$  and a line  $AB$  intersecting it are shown in Fig. 164. If a plane  $\beta$  is drawn through line  $AB$  and the line  $MN$  is the line of intersection of planes  $\alpha$  and  $\beta$ , then the point  $K$  at the intersection of lines  $AB$  and  $MN$  will be the point in which line  $AB$  pierces plane  $\alpha$ . This is so because line  $MN$  is the locus of points common to both planes  $\alpha$  and  $\beta$ , and point  $K$ , at the intersection of lines  $AB$  and  $MN$  which lie in these planes, is a point common to line  $AB$  and plane  $\alpha$ .

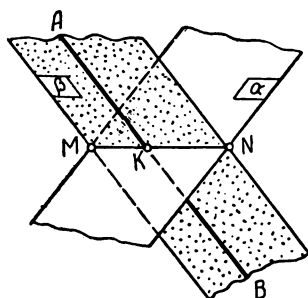


Fig. 164

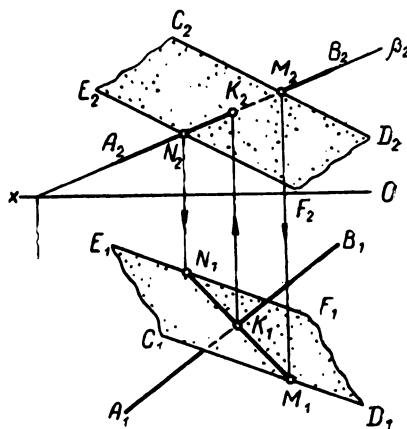


Fig. 165

The construction of the piercing point of a line in a plane generally consists of three operations:

1. The construction of an auxiliary plane through the line.
2. The determination of the line of intersection of the given plane with the auxiliary plane.
3. Finally, the determination of the point of intersection of the given line with the constructed line of intersection of the two planes.

Projecting planes are generally used as auxiliary planes because of the simplicity with which these planes, passing through straight lines, may be shown on the drawing, due to the collecting properties of the traces of such planes.

A line  $AB$  and a plane  $\alpha$ , given by parallel lines  $CD$  and  $EF$ , are shown in Fig. 165. It is required to find the meeting point of line  $AB$  with plane  $\alpha$ . The following operations are necessary:

1. Draw an auxiliary frontal-projecting plane  $\beta$  through line  $AB$ . Its frontal trace-projection merges with the frontal projection



$A_2B_2$  of line  $AB$ . Its horizontal trace is perpendicular to the ground line  $Ox$ .

2. Construct the line  $MN$  of intersection of planes  $\alpha$  and  $\beta$ . To do this find points  $M$  and  $N$  in which lines  $CD$  and  $EF$  pierce the auxiliary plane.

3. The intersection of lines  $AB$  and  $MN$  gives the required piercing point  $K$  of line  $AB$  in plane  $\alpha$ . The horizontal projection  $K_1$  of this point lies at the intersection of projections  $A_1B_1$  and  $M_1N_1$ . Draw a projector from point  $K_1$  to obtain the frontal projection  $K_2$  of point  $K$  contained in the frontal projection  $A_2B_2$  of line  $AB$ .

The construction of the piercing point of line  $AB$  in plane

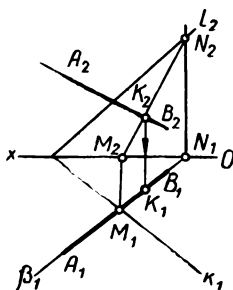


Fig. 166

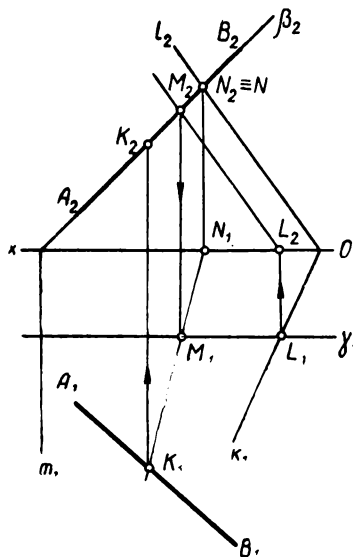


Fig. 167

$\alpha$ , given by traces  $k_1$  and  $l_2$ , is shown in Fig. 166. The auxiliary plane  $\beta$  passed through line  $AB$  is perpendicular to the plane of projection  $\Pi_1$ . Trace  $\beta_1$  coincides with projection  $A_1B_1$ . The line  $MN$  is the line of intersection of planes  $\alpha$  and  $\beta$ . The intersection of projections  $A_2B_2$  and  $M_2N_2$  determines the frontal projection  $K_2$  of point  $K$ , which is the piercing point of line  $AB$  in plane  $\alpha$ . The horizontal projection  $K_1$  of point  $K$  is found by drawing through  $K_2$  a projector to intersect line  $A_1B_1$ .

2. When determining the piercing of a line with a plane, the traces of which are given, it may happen that one pair of corresponding traces of the given and the auxiliary plane do not intersect within the limits of the drawing. In that case it is necessary to introduce a second auxiliary cutting plane (see Fig. 151). Such a case is considered in Fig. 167.

When an auxiliary frontal-projecting plane  $\beta$  is drawn through line  $AB$  and the frontal trace  $N$  of the line of intersection of planes

$\alpha$  and  $\beta$  has been determined, it may be seen that the horizontal trace  $m$  of plane  $\beta$  does not intersect the horizontal trace  $k_1$  of plane  $\alpha$  within the limits of the drawing. So in order to find the second point  $M$  contained in the line  $MN$  of intersection of planes  $\alpha$  and  $\beta$ , draw an auxiliary plane  $\gamma$  parallel to  $\Pi_2$ . The intersections of plane  $\gamma$  with  $\alpha$  and  $\beta$  will be frontals. The projection on plane  $\Pi_2$  of the frontal of plane  $\alpha$  will pass through point  $L_2$  and lie parallel to trace  $l_2$ , while the frontal projection of the frontal plane  $\beta$  will coincide with trace  $\beta_2$  which possesses collecting properties. The intersection of the frontal projections of these frontals gives the frontal projection  $M_2$  of point  $M$ . The horizontal projection  $M_1$  of point  $M$  will lie at the intersection of projector  $M_2M_1$  with trace  $\gamma_1$ . This trace possesses collecting properties. The line of intersection of planes  $\alpha$  and  $\beta$  passes through points  $M$  and  $N$ . This determines the piercing point  $K$  of the line  $AB$  in plane  $\alpha$ .

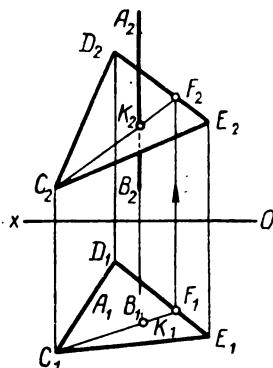


Fig. 168

3. When the line piercing a given plane is perpendicular to the plane of projection the projection of the point of intersection on the plane to which the line is perpendicular coincides with the projection of the point on that same plane. So the problem may be reduced to finding the second projection of the point of intersection of the line with a plane by means of one known projection of the point.

This case is illustrated in Fig. 168, where the line  $AB$  is perpendicular to plane  $\Pi_1$  and pierces the plane of triangle  $CDE$ . The horizontal projection  $K_1$  of the piercing point  $K$  in plane  $CDE$  coincides with point  $A_1 \equiv B_1$  which is the projection of line  $AB$  on plane  $\Pi_1$ .

To construct the frontal projection  $K_2$  of piercing point  $K$  through point  $K_1$  draw the auxiliary line  $C_1K_1F_1$  and, assuming it to be the horizontal projection of line  $CKF$  contained in the plane of triangle  $CDE$ , by means of projection  $C_1F_1$  determine the frontal projection  $C_2F_2$  of the line. The frontal projection  $K_2$  of the piercing point of line  $AB$  in the plane of the triangle  $CDE$  lies at the intersection of lines  $C_2F_2$  and  $A_2B_2$ .

4. The piercing point of a line with a projecting plane is determined directly without auxiliary constructions.

For instance, in Fig. 169,  $a$  the horizontal projection  $K_1$  of the piercing point  $K$  of line  $AB$  in the horizontal-projecting plane  $\alpha$ , which is given by the triangle  $CDE$ , lies at the intersection of the

projection-trace of that plane on  $\Pi_1$  with the projection of the line  $AB$  on the same plane of projection  $\Pi_1$ . The frontal projection  $K_2$  of point  $K$  is found by passing a projector through point  $K_1$ . In Fig. 169,  $b$  the projections  $K_2$  and  $K_1$  of piercing point  $K$  of line  $AB$  in the frontal-projecting plane  $\beta$  are shown.

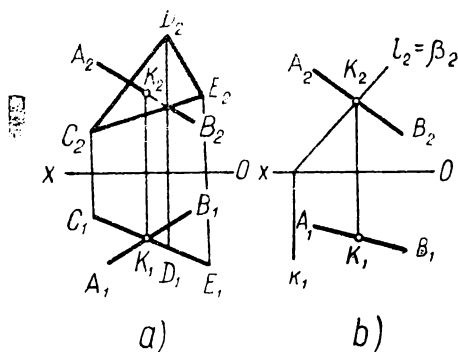


Fig. 169

5. Next let us examine the constructions required to determine the piercing point of a line with a plane by means of the method of oblique projections.

The construction of oblique projections in the system of planes of projection  $\Pi_1$  and  $\Pi_2$  is similar to the construction of traces of lines and planes (§ 11 and 15, 7). The straight line  $AB$  and plane  $\alpha$  which is given by the triangle  $CDE$ , are shown in Fig. 170. It is required to determine the piercing point of line  $AB$  in plane  $\alpha$ .

Project the plane and line in a direction parallel to the horizontal  $EF$  of triangle  $CDE$ . The oblique projections of the vertices of the triangle  $C$ ,  $D$  and  $E$  will be found as traces of horizontals drawn in the plane of the triangle through these points. The oblique projections  $\bar{C}_1$ ,  $\bar{D}_1$  and  $\bar{E}_1$  (read as "C line prime," etc.) lie on ground line  $Ox$  and the projections  $\bar{C}_2$ ,  $\bar{D}_2$  and  $\bar{E}_2$  lie on the frontal trace  $l_2$  of plane  $\alpha$ .

By drawing horizontal lines parallel to the horizontal  $EF$  through points  $A$  and  $B$  and having found the traces of these lines we obtain oblique projections  $\bar{A}_1$ ,  $\bar{A}_2$  and  $\bar{B}_1$ ,  $\bar{B}_2$  of points  $A$  and  $B$ . The oblique projection  $\bar{A}_1\bar{B}_1$  of line  $AB$  coincides with ground line  $Ox$  and its oblique projection  $\bar{A}_2\bar{B}_2$  will cut the oblique projection of triangle  $\bar{C}_2\bar{D}_2\bar{E}_2$ , i. e., trace  $l_2$  in point  $\bar{K}_2$ .

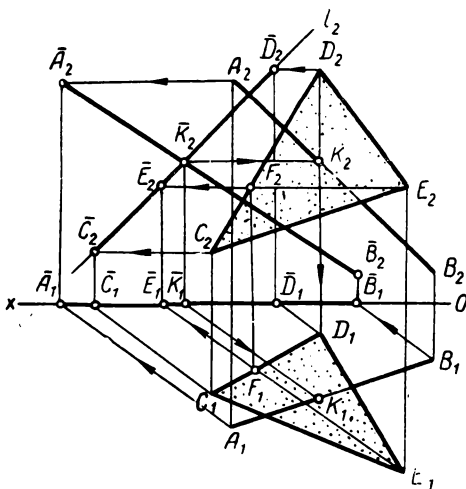


Fig. 170

The point  $\bar{K}_2$  is the oblique projection on  $\Pi_2$  of point  $K$  which is the piercing point of line  $AB$  in the plane of triangle  $CDE$ . The oblique projection  $K_1$  of this point lies on ground line  $Ox$ .

By carrying out the reverse projection, namely by drawing through points  $K_1$  and  $K_2$  the projections of a line parallel to the horizontal  $EF$ , at their intersections with the corresponding projections of line  $AB$  we find the projections  $K_1$  and  $K_2$  of  $K$ , that is the piercing point of line  $AB$  in the plane of triangle  $CDE$ .

### Problems and Exercises

1. Which planes are generally used as auxiliary planes when determining the piercing point of a given line in a plane?
2. What constructions are required to determine the piercing point of a line in a plane?
3. When is the piercing point of a line in a plane determined directly, without auxiliary constructions? How is this done?
4. How is the piercing point of a line in a plane found by oblique projection?
5. Find the piercing point of line  $AB$  in a plane given by the intersecting lines  $CD$  and  $DE$  (Fig. 171).

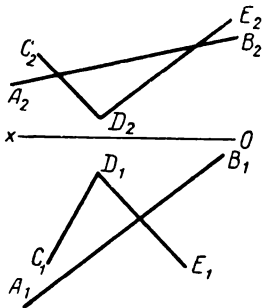


Fig. 171

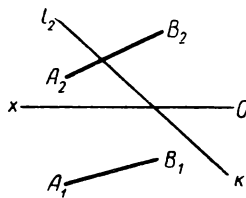


Fig. 172

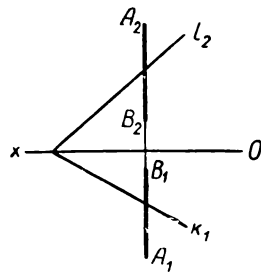


Fig. 173

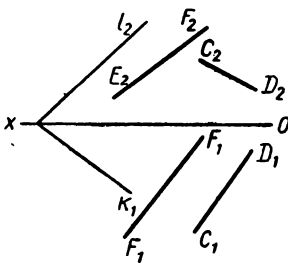


Fig. 174

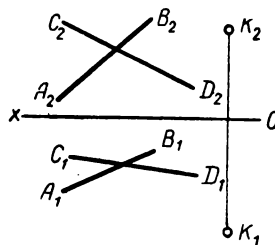


Fig. 175



the horizontal (marked by the letters *h.p.h.* in Fig. 177, *b*) and the frontal projection of the frontal (*f.p.f.*) of a plane are parallel to the corresponding traces of the plane, it follows that in the multiview drawing the projections of a line perpendicular to a plane, are perpendicular to the corresponding traces of the plane and also to corresponding projections of any horizontal and frontal of the plane.

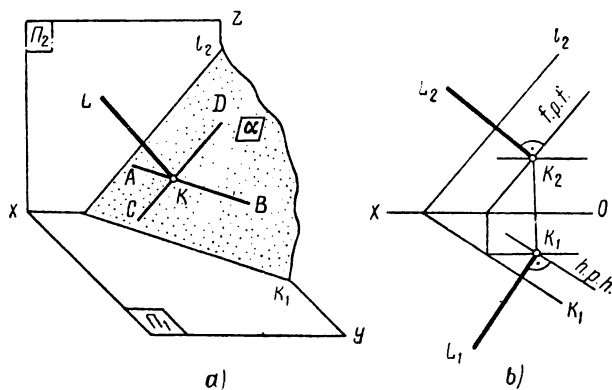


Fig. 177

This being so the construction of a perpendicular to the plane  $\alpha$ , given by triangle  $BCD$  (Fig. 178) and passing through a point  $A$ , consists of the following: 1) the construction of a horizontal  $DE$  and a frontal  $BF$  of the plane, and 2) the drawing of a perpendicular from  $A_1$  to  $D_1E_1$  and another from  $A_2$  to  $B_2F_2$ , i. e.,  $A_1K_1 \perp D_1E_1$  and  $A_2K_2 \perp B_2F_2$ .

2. The determination of the distance of a point from a plane. The problem of determining the distance of a point from a plane may be subdivided as follows: 1) the construction of a perpendicular to the plane from a given point; 2) the determination of the piercing point of this perpendicular in the plane; 3) the determination of the true length of the perpendicular line the length of which is the distance required.

It is required to determine the distance of point  $D$  from the plane of triangle  $ABC$  (Fig. 179).

1. In the plane of triangle  $ABC$  draw the horizontal  $AL$  and frontal  $CF$ . From point  $D_1$  draw a perpendicular to the horizontal projection  $A_1L_1$  of the horizontal and from point  $D_2$  draw a perpendicular to the frontal projection  $C_2F_2$  of the frontal. Line  $DE$  is then the perpendicular drawn from point  $D$  to the plane  $ABC$ .

2. The piercing point of perpendicular  $DE$  with plane  $ABC$  is found with the help of a frontal-projecting plane  $\gamma$  passed through  $DE$ . The frontal trace  $\gamma_2$  of this plane coincides with the frontal projection  $D_2E_2$  of the perpendicular  $DE$ . The horizontal trace  $m_1$  is perpendicular to ground line  $Ox$ . Find the line  $MN$  of intersection of planes  $ABC$  and  $\gamma$ . The intersection of lines  $DE$  and  $MN$  gives

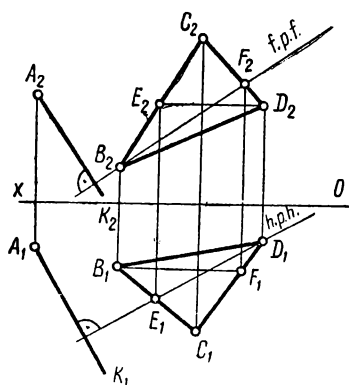


Fig. 178

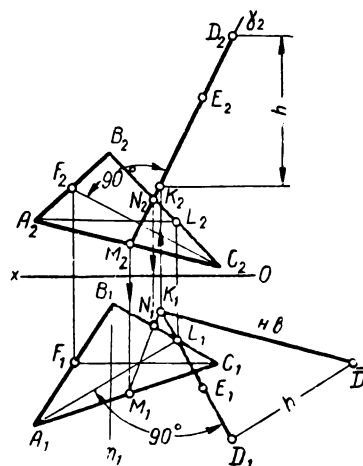


Fig. 179

point  $K$  where the perpendicular  $DE$  pierces the plane  $ABC$ . The length of line  $DK$ , whose projections  $D_1K_1$  and  $D_2K_2$  have been found, is the distance from point  $D$  to plane  $ABC$ .

3. The true length of this line  $DK$  is found by the right triangle method. Having obtained the difference  $h$  of the distances of points  $D$  and  $K$  from plane  $\Pi_1$ , construct the right triangle  $D_1K_1\bar{D}$ , one side of which is the horizontal projection  $D_1K_1$  of line  $DK$ . A second side is equal in length to  $h$ . The hypotenuse  $K_1\bar{D}$  gives the true value of the required distance from point  $D$  to the plane of triangle  $ABC$ .

3. If the plane is one of the projecting planes, as in Fig. 180, where the plane  $\alpha$  is a frontal-projecting plane, the determination of the distance of the plane from the given point  $A$  is considerably simplified. This is so because the frontal projection  $K_2$  of the piercing point  $K$  of the perpendicular drawn from point  $A$  to plane  $\alpha$  lies on the trace  $l_2$ , while the perpendicular  $AK$  is parallel to plane  $\Pi_2$  and so is projected onto  $\Pi_2$  in its true length.

4. In order to pass a plane horizontal to line  $BC$  through a point  $A$ , a horizontal or frontal at right angles to the given line is constructed (Fig. 181). By drawing through point  $A$  the horizontal  $AN$ , whose horizontal projection  $A_1N_1$  is

perpendicular to  $B_1C_1$  and, having found trace  $N$  of this horizontal, draw through point  $N_2 \equiv N$  the trace  $l_2 \perp B_2C_2$ . Through the point of convergence of the traces draw the trace  $k_1 \perp B_1C_1$ . The plane  $\alpha$  so constructed is the required plane, since it passes through point  $A$  and is perpendicular to line  $BC$ , moreover its traces  $k_1$  and  $l_2$  are perpendicular to the corresponding projections of the given line.

5. The problem of determining the distance of a point from a line may be solved with the help of a plane passed through the point perpendicular to the line.

The solution of this problem is as follows (Fig. 182):

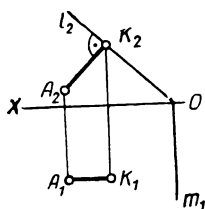


Fig. 180

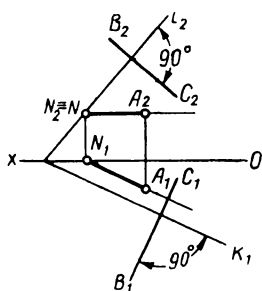


Fig. 181

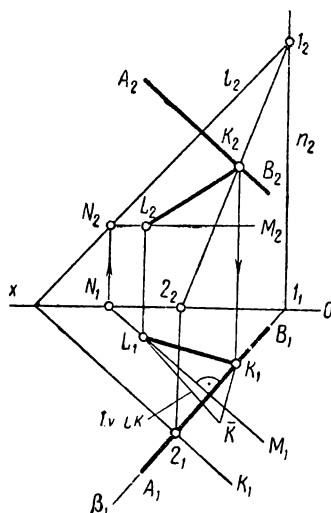


Fig. 182

1. Construct the plane  $\alpha \perp AB$  with the help of the horizontal  $MN$  drawn through point  $L$  perpendicularly to  $AB$  (see Fig. 181).
2. Find the meeting point  $K$  of line  $AB$  with plane  $\alpha$  using plane  $\beta \perp \Pi_1$  passed through  $AB$ .

3. Join up points  $K$  and  $L$ . The perpendicular  $KL$  located in the plane  $\alpha \perp AB$  enables the distance of point  $L$  from line  $AB$  to be determined.

4. The true length  $\bar{KL}_1$  of line  $KL$  is found by constructing the right-angle triangle  $K_1L_1\bar{K}$ .

## § 22. Perpendicular Planes

1. Two planes are mutually perpendicular if in each plane a line may be drawn perpendicular to the other.

It follows from this that, in order to pass a plane perpendicular to the triangle  $ABC$  through some point  $M$  (Fig. 183), the first step is to construct a line  $MN$  perpendicular to the given plane. To do so the horizontal  $AL$  and



the frontal  $CF$  are drawn. The horizontal projection  $M_1N_1$  of the perpendicular passing through  $M$  will be perpendicular to the horizontal projection  $A_1L_1$  of the horizontal  $AL$ . The frontal projection  $M_2N_2$  of the perpendicular will, in turn, be perpendicular to the frontal projections  $C_2F_2$  of the frontal  $CF$ . A plane passing through line  $MN$  will be perpendicular to the plane of triangle  $ABC$ .

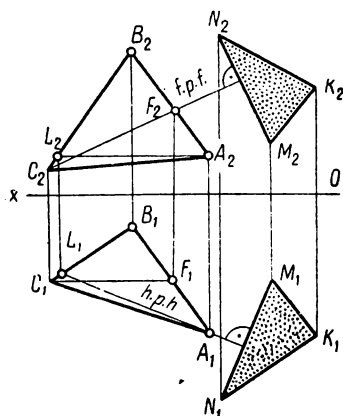


Fig. 183

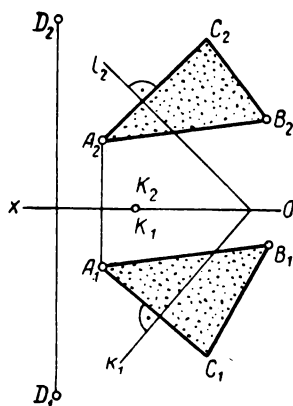


Fig. 184

An infinite number of planes perpendicular to plane  $ABC$  could be drawn through line  $MN$ . The plane determined by triangle  $MNK$  in Fig. 183 is one of them.

In Fig. 184 the plane of triangle  $ABC$  is perpendicular to plane  $\alpha$  given by traces  $k_1$  and  $l_2$ , since the side  $AC$  of this triangle is perpendicular to plane  $\alpha$  ( $A_1C_1 \perp k_1$  and  $A_2C_2 \perp l_2$ ).

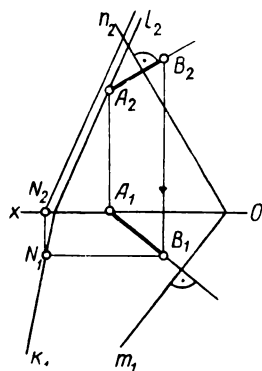


Fig. 185

2. The problem whether two given planes are perpendicular to each other is solved by drawing through a point taken in one of these planes a line perpendicular to the other plane. If that perpendicular is contained in the first plane, then the planes are perpendicular to each other.

The planes  $\alpha$  and  $\beta$ , depicted in Fig. 185, are not perpendicular, since the perpendicular  $AB$  drawn to plane  $\beta$  from a point  $A$ , taken on trace  $l_2$  of plane  $\alpha$ , does not lie in plane  $\alpha$ . This may be proved by drawing an auxiliary line in plane  $\alpha$ . Such a line is the frontal  $BN$  in Fig. 185.

It is clear that frontal  $BN$  does not intersect the perpendicular  $AB$  and is not parallel to it.

3. Planes  $\alpha$  and  $\beta$  whose corresponding traces are perpendicular, i. e.,  $k_1 \perp m_1$  and  $l_2 \perp n_2$ , are not perpendicular to each other. This is so because a

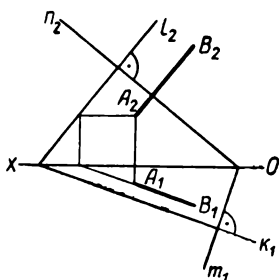


Fig. 186

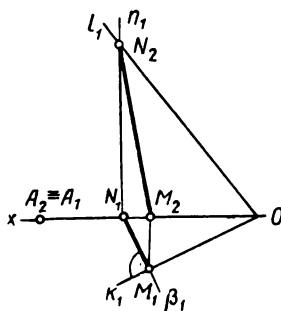


Fig. 187

perpendicular  $AB$  drawn to plane  $\beta$  from point  $A$ , taken in plane  $\alpha$ , is not contained in plane  $\alpha$ , since its projections are parallel to the corresponding traces of this plane ( $A_1B_1 \parallel k_1$  and  $A_2B_2 \parallel l_2$ ).

4. Attention should be paid to the fact that the mutually perpendicular disposition of corresponding traces of two oblique lines, such as those in Fig. 186, is a sign that such planes do not intersect at right angles.

If at least one of two intersecting planes is a projecting plane and if the angle formed by the trace-projection of the projecting plane and the similar trace of the other plane is a right angle, then the planes are mutually perpendicular.

Such a case is shown in Fig. 187. The oblique plane is perpendicular to the horizontal-projecting plane  $\beta$  since the horizontal trace  $k_1$  of plane  $\alpha$  is perpendicular to plane  $\beta$ . This is so because  $k_1 \perp \beta_1$  and the frontal trace  $n_1$  is perpendicular to the ground line  $Ox$  with which the frontal projection of trace  $k$  ( $k_2 \equiv Ox$ ) coincides.

If one of two mutually perpendicular planes is a profile plane and the other is parallel to ground line  $Ox$  or passes through it, then the corresponding traces of such planes in the  $\Pi_1-\Pi_2$  system of planes of projection are mutually perpendicular.

It should be noted that the line of intersection  $MN$  of the mutually perpendicular planes  $\alpha$  and  $\beta$  (Fig. 187) is perpendicular to trace  $k_1$ , since  $M_1N_1 \perp k_1$ . Therefore  $MN$  is at the same time the line of maximum inclination of plane  $\alpha$  to the horizontal plane of projections  $\Pi_1$ . It follows that the line of maximum inclination of

a given plane may be constructed by intersecting it by a projecting plane perpendicular to the trace of the given plane (for comparison see § 17).

## Problems and Exercises

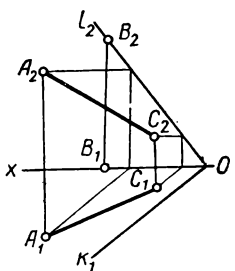
1. What are the positions in the multiview drawing of the projections of a line which is perpendicular to a plane given by its traces?

2. How is the true distance to be found of a point from a) a horizontal-projecting plane, b) an oblique plane?

3. How are the frontal traces of two mutually perpendicular planes seen in the multiview drawing if one of them is an oblique plane and the other is a frontal-projecting plane?

4. Are oblique lines perpendicular to each other if their corresponding traces in  $\Pi_1$  and  $\Pi_2$  are mutually perpendicular?

5. What planes should be made to intersect a plane in order to obtain in it a line of maximum inclination a) to plane  $\Pi_1$ , b) to plane  $\Pi_2$ , and c) to plane  $\Pi_3$ ?



*Fig. 188*

6. Find the distance of point  $D$  from the plane of triangle  $ABC$  (Fig. 184).

7. Find the distance of point  $K$  contained in ground line  $Ox$  from plane  $\alpha$  (Fig. 184).

8. Determine whether the planes given by triangles  $ABC$  and  $DEF$  are perpendicular (Fig. 197).

Hint: From any point taken in one of the planes drop a perpendicular to the other plane and check whether the perpendicular belongs to the first plane (see Fig. 185 and text).

9. Through a point  $C$  taken on line  $AB$  draw a line  $CL$  perpendicular to  $AB$  to intersect line  $DE$ .

Hint. With the help of level lines perpendicular to  $AB$  pass through point  $C$  a plane perpendicular to  $AB$ . Find the point of intersection  $L$  of line  $DE$  with the plane drawn and join points  $C$  and  $L$ .

10. Construct a rectangle  $ABCD$  by means of two projections of its side  $AB$  and the horizontal projection of side  $BC$ .

**Hint.** With the help of level lines perpendicular to  $AB$  draw an auxiliary plane perpendicular to  $AB$  and passing through point  $B$ . The vertex  $C_1$  will lie in this plane. Having found projection  $C_1$  find projection  $C_2$  and join up points  $C_2$  and  $B_2$ .

The projections of the other two sides of the rectangle are parallel and equal to the corresponding projections of its sides  $AB$  and  $BC$ .

11. In plane  $\alpha$  construct a parallelogram  $ABCD$  given its diagonal  $AC$  and its vertex  $B$  (Fig. 188).

12. By means of the frontal projection  $C_2D_2$  construct the horizontal projection  $C_1D_1$  of a line  $CD$  which is to be perpendicular to a given line  $AB$ .

Hint. Draw a plane perpendicular to line  $AB$  and in it construct an auxiliary line parallel to line  $CD$ . The required projection  $C_1D_1$  will be parallel to the horizontal projection of this auxiliary line.

13. Through the piercing point of line  $AB$  in an oblique plane  $\alpha$  draw a line perpendicular to  $AB$  lying in the plane.

14. Construct a plane  $\beta$ , parallel to a given plane  $\alpha$  and situated 20 mm from it.

### § 23. The Determination of Visibility of Geometrical Elements in Multiview Drawings

1. In order that a drawing may be as clear as possible the visible contours of the object represented are usually shown by continuous lines and hidden lines, by broken lines. Sometimes invisible parts are not shown at all.

The methods of determining the visibility of elements in the multiview drawing are given below.

Of two given points lying on a line perpendicular to a given plane of projection, the point nearer to the eye of the observer will be the visible point. It is assumed that the observer is standing in the first octant looking along the line perpendicular to the plane of projection to which the two points belong.

It follows that, of two points lying on a line perpendicular to a plane of projection, the one farther away from the plane is the visible point.

The points  $A$  and  $B$  (Fig. 189,  $a$ ) are situated on the same projector perpendicular to plane  $\Pi_2$ . Point  $A$  being farther away from plane  $\Pi_2$  covers point  $B$ . So, when projected onto  $\Pi_2$ , point  $A$  is visible and point  $B$  is hidden. When projected onto plane  $\Pi_1$  both points  $A$  and  $B$  are visible because they lie on different lines perpendicular to plane  $\Pi_1$ .

In the multiview drawing (Fig. 189,  $b$ ) the frontal projections of points  $A$  and  $B$  coincide,  $A_2 \equiv (B_2)$ , since these points in space lie on the same frontal projector. The horizontal projection  $A_1$  of point  $A$  lies farther away from ground line  $Ox$  than the horizontal projection  $B_1$  of point  $B$ , so it follows that point  $A$  is farther away from plane  $\Pi_2$  and, being projected on this plane  $\Pi_2$ , covers point  $B$ .

Points  $C$  and  $D$  (Fig. 189,  $a$ ) lie on the same line perpendicular to plane  $\Pi_1$ , point  $C$  being higher than point  $D$ , that is to say, farther away from plane  $\Pi_1$ , so point  $C$ , when projected onto plane  $\Pi_1$ , will be visible and point  $D$  will be hidden.

In the multiview drawing, Fig. 189, *b*, the horizontal projections of points *C* and *D* coincide,  $C_1 \equiv (D_1)$ . The frontal projection  $C_2$  of point *C* is farther from ground line *Ox* than the frontal projection  $D_2$  of point *D*, so point *C* is farther from plane  $\Pi_1$  and, being projected onto that plane, will hide point *D*.

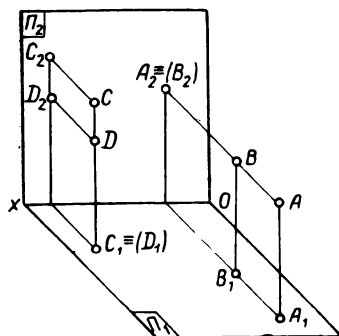


Fig. 189

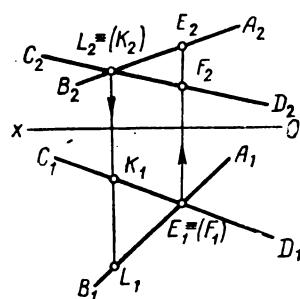
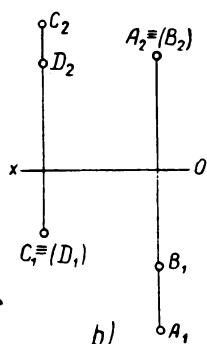


Fig. 190

2. Visibility of points on non-intersecting lines.

Two skew lines *AB* and *CD* are shown in Fig. 190. Point *E*, contained in line *AB*, and point *F*, contained in line *CD*, are situated on the same line perpendicular to plane  $\Pi_1$ . The horizontal projections of points *E* and *F* merge as one point, where the horizontal projections of the given lines *AB* and *CD* cross. Point *E* is farther from plane  $\Pi_1$  than point *F*, and so, when projected on plane  $\Pi_1$ , will be visible.

Point *L*, contained in line *AB*, and point *K*, contained in line *CD*, are situated on one perpendicular to plane  $\Pi_2$ . The frontal projections of points *L* and *K* merge into the point, where the frontal projections of the given lines *AB* and *CD* cross.

Point *L* lies farther away from plane  $\Pi_2$  than point *K* and, when projected onto plane  $\Pi_2$ , will be visible.

3. Next let us consider the visibility of the parts of the projections of a line *MN* intersecting an opaque triangular plate *ABC* (Fig. 191).

By passing through *MN* an auxiliary frontal-projecting plane, point *K*, where line *MN* pierces triangular plate *ABC*, is found. This point *K* is the boundary of visibility of the parts of line *MN*. Let us determine the visibility of separate parts of line *MN* when it is projected onto plane  $\Pi_2$ . Take a point *E* on the side *AB* of the triangle and a point *F* on the line *MN*, so that points *E* and *F*

project onto plane  $\Pi_2$  as one point  $E_2 \equiv F_2$ . The horizontal projection  $E_1$  of point  $E$  lies farther away from ground line  $Ox$  than the horizontal projection  $F_1$  of point  $F$ . In other words, the value of  $y$  for point  $E$  is greater than that of  $y$  for point  $F$ . Therefore point  $E$ , when it is projected onto plane  $\Pi_2$ , hides point  $F$ .

The part  $FK$  of line  $MN$  when projected onto plane  $\Pi_2$  will not be visible. It is hidden by the opaque triangular

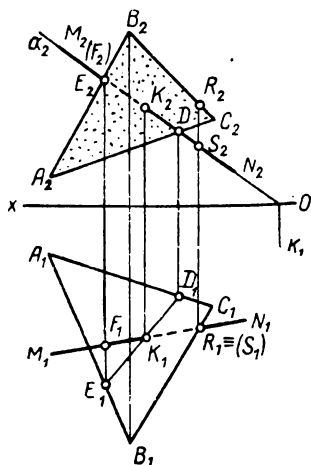


Fig. 191

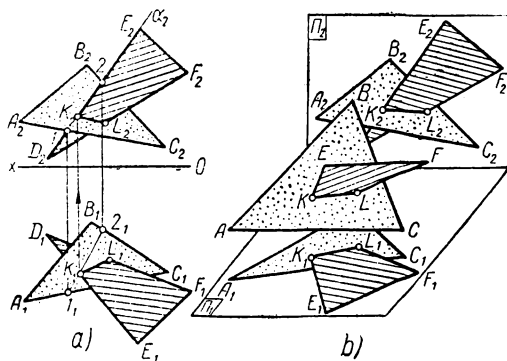


Fig. 192

plate  $ABC$ , whereas part  $KN$  remains visible since it lies in front of the plate.

The visibility of parts of line  $MN$ , when projected onto plane  $\Pi_1$ , may be determined by comparing the values of  $z$  for points  $R$  and  $S$ . The horizontal projections of these points merge into one point  $R_1 \equiv S_1$  and are located at the intersection of lines  $M_1N_1$  and  $B_1C_1$ . Since the distance  $z$  of point  $R$ , which belongs to side  $BC$  of the triangle, is greater than that of point  $S$  which belongs to line  $MN$ , the part  $KS$  of line  $MN$  lies below the triangular plate  $ABC$  and so, when projected onto plane  $\Pi_1$ , is invisible. The part  $SN$  of line  $MN$  when projected on plane  $\Pi_1$  is visible since it is not covered by the plate  $ABC$ .

4. Now let us find the line of intersection of the opaque triangular plates  $ABC$  and  $DEF$ . The visible part of triangle  $ABC$  will be shown by dots and that of triangle  $DEF$  by hatching (Fig. 192).

To construct the line of intersection  $KL$  of these triangles find the piercing point  $K$  of side  $ED$  with triangle  $ABC$ . This is done with the help of the frontal-projecting plane  $\alpha$  and point  $L$  in which side  $FD$  pierces triangle  $ABC$ . This construction is not shown in Fig. 192. Next join points  $K$  and  $L$ .

The visibility of the separate parts of the triangles  $ABC$  and  $DEF$  when projected onto  $\Pi_1$  and  $\Pi_2$  may be established by determining the visible parts of the sides  $ED$  and  $FD$ , as was done with line  $MN$  in Fig. 191. The visible parts of triangles  $ABC$  and  $DEF$  are shown by dots and hatching respectively. The parts of the sides of the triangle that are hidden are not shown. The space drawing of the intersection of the triangles  $ABC$  and  $DEF$  and the projections of these triangles on planes  $\Pi_1$  and  $\Pi_2$  are shown in Fig. 192, *b*.

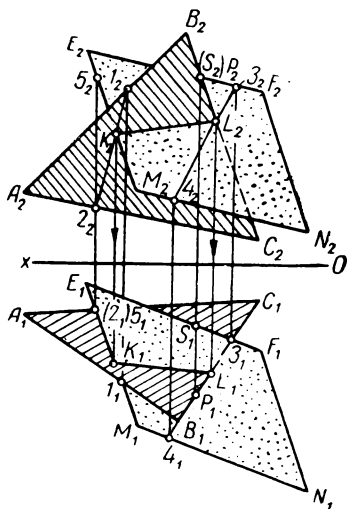


Fig. 193

5. Let us construct the line of intersection of triangle  $ABC$  with parallelogram  $EFNM$ , both of which are assumed to be opaque plates (Fig. 193).

In order to determine the line of intersection  $KL$  of the given figures, first find point  $K$  in which the side  $EM$  of the parallelogram pierces the plane of the triangle, and point  $L$ , where side  $BC$  of the triangle pierces the plane of the parallelogram. Points  $K$  and  $L$  are then joined up.

The visibility of the various parts, when projected onto plane  $\Pi_2$ , is determined with the help of points  $S$  and  $P$  taken on side  $EF$  of the parallelogram and on the side  $BC$  of the triangle, respectively. By comparing the values of  $y$  for points  $S$  and  $P$  it can be seen that point  $P$  is situated farther away from plane  $\Pi_2$  than point  $S$ . It follows that part  $BL$  of side  $BC$  of the triangle will be visible when projected onto plane  $\Pi_2$ . Beyond the boundary of visibility, point  $L$ , the side  $BC$  is covered by the parallelogram and again becomes visible in the vicinity of the vertex  $C$  since this part is not hidden by the plane of the parallelogram.

The visibility of the various parts, when projected onto plane  $\Pi_1$ , is established by comparing the values of  $z$  for points  $2$  and  $5$  taken on sides  $AC$  and  $EM$  of the given figures. The value of  $z$  for point  $2$  is smaller than that of point  $5$ . It follows that point  $5$ , together with the corresponding part of side  $EM$  of the parallelogram up to point  $K$ , when projected onto plane  $\Pi_1$ , is visible. Beyond point  $K$  the side  $EM$  passes under the triangle  $ABC$  and again becomes visible when it emerges from the plane of triangle  $ABC$ .

## Problems and Exercises

1. If two points are situated on one perpendicular to a plane of projections, how is their visibility determined?

2. When will a line which lies behind an opaque plate and is projected onto plane  $\Pi_1$ , be visible if it is projected onto plane  $\Pi_2$ ?

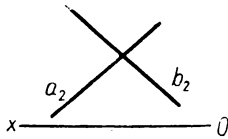


Fig. 194

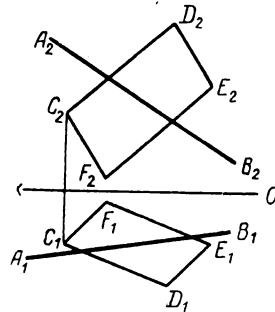


Fig. 195

3. On the two non-intersecting lines  $a$  and  $b$  find the points, which are not visible when projected a) onto  $\Pi_1$ , b) onto  $\Pi_2$ , c) onto  $\Pi_3$  (Fig. 194).

4. Determine the visible and hidden parts of line  $AB$  when it pierces the plane of parallelogram  $CDEF$  (Fig. 195).

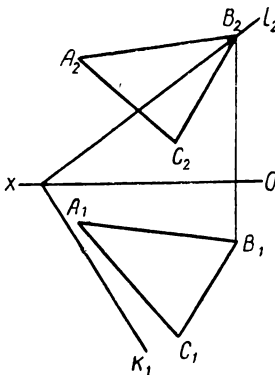


Fig. 196

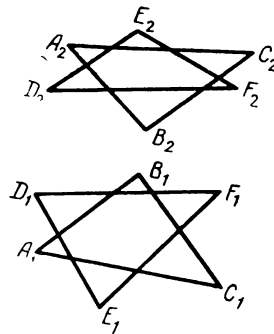


Fig. 197

5. Construct the line of intersection of plane  $\alpha$ , given by traces  $k_1$  and  $l_2$ , with the triangle  $ABC$ . Show its visible part by hatching (Fig. 196).

6. Construct the line of intersection of triangles  $ABC$  and  $DEF$ , showing visible parts of the triangle  $ABC$  by hatching and those of the triangle  $DEF$  by dots (Fig. 197).



## Chapter VI

### THE TRANSFORMATION OF PROJECTIONS

If straight lines or plane figures are parallel or perpendicular to the planes of projection, then distances, angles and the relative positions of individual geometrical elements in space, as seen in the multiview drawings, may be measured without any additional construction. That is to say true views are obtained. For example, the true distance of a point from a line is given by the projection on the horizontal plane  $\Pi_1$  if the given line is perpendicular to that plane. Any figure lying in a horizontal plane will also be projected onto plane  $\Pi_1$  in its true size.

The determination of the true dimensions of oblique lines and planes requires special constructions transforming the projections into others enabling the true views to be determined directly.

Any course of descriptive geometry must outline various methods of transforming projections. These are the substitution of one or more planes of projection by others, the method of rotating the planes of projection and the method of coincidence.

#### § 24. Substituting Planes of Projection

1. A point  $A$  is given relative to the pair of planes of projection  $\Pi_1$ - $\Pi_2$  (Fig. 198, *a*). Instead of the frontal plane  $\Pi_2$  an additional plane of projection  $\Pi_4$ , perpendicular to plane  $\Pi_1$ , may be introduced. In the case of such a substitution the profile plane of projection in the principal system of projection planes  $\Pi_1$ - $\Pi_2$ - $\Pi_3$  continues to be referred to as  $\Pi_3$ . The planes  $\Pi_1$  and  $\Pi_4$  constitute a new pair of mutually perpendicular planes of projection and the line  $s_{14}$  will be the new ground line.

Let us construct the orthographic projections of point  $A$  in the new system of planes of projection  $\Pi_1$ - $\Pi_4$ . The horizontal projection  $A_1$  of point  $A$  in the new system  $\Pi_1$ - $\Pi_4$  will still be the horizontal projection of point  $A$  since plane  $\Pi_1$  remains unchanged. To obtain the projection  $A_4$  of point  $A$  on plane  $\Pi_4$  draw a perpendicular to that plane from point  $A$  together with the rectangle  $AA_1A_{14}A_4$ , of which one side is the line  $AA_1$  and

another the perpendicular  $A_1A_{14}$  drawn from point  $A_1$  to ground line  $s_{14}$ .

Since  $A_4A_{14}$  and  $AA_1$  are the opposite sides of rectangle  $AA_1A_{14}A_4$ , they are equal. But  $AA_1 = A_2A_{12}$ , so  $A_4A_{14} = A_2A_{12}$ . In other words, the distance of the new projection  $A_4$  from the new ground line  $s_{14}$  is equal to the

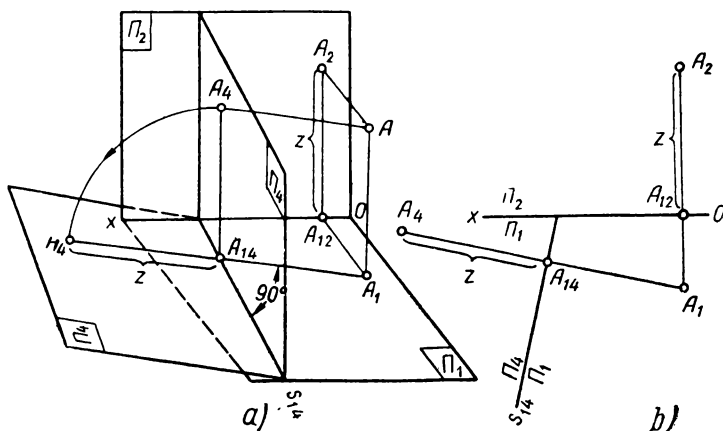


Fig. 198

distance of the original projection  $A_2$  from ground line  $Ox$ .

In order to pass from the three-dimensional representation to the multiview drawing plane  $\Pi_4$  is rotated about ground line  $s_{14}$  to coincide with plane  $\Pi_1$ . Projection  $A_4$  of point  $A$  will then lie on the same perpendicular to the ground line  $s_{14}$  as point  $A_1$ .

Planes  $\Pi_1$  and  $\Pi_4$  may be brought into coincidence by rotating  $\Pi_4$  in either direction. However, the direction of rotation should be such that the projections  $A_1$  and  $A_4$  of point  $A$  lie on opposite sides of ground line  $s_{14}$ .

The construction of projection  $A_4$  of point  $A$  in the multiview drawing (Fig. 198, b) is carried out in the following order: 1) draw ground line  $s_{14}$  of the new system of planes of projection  $\Pi_1-\Pi_4$ ; 2) from point  $A_1$  draw a perpendicular to the line  $s_{14}$ ; 3) on this perpendicular from the line  $s_{14}$  lay off distance  $A_4A_{14}$  equal to the length of line  $A_2A_{12}$ . The point  $A_4$  obtained in this way is the orthographic projection of point  $A$  on plane  $\Pi_4$ .

2. The construction of the multiview drawing of point  $B$  (Fig. 199, a), when the horizontal plane of projection  $\Pi_1$  is changed for the new plane  $\Pi_4$ , also perpendicular to plane  $\Pi_2$ , is carried out in the same sequence. First, the ground line  $s_{24}$  of the  $\Pi_2-\Pi_4$  system of planes of projection is drawn. Then, from point  $B_2$  a

perpendicular is drawn to line  $s_{24}$ . Finally, on this perpendicular the distance  $B_4B_{24}=B_1B_{12}$  is laid off from line  $s_{24}$ . In this case the distance of projection  $B_4$  of point  $B$  from the new ground line  $s_{24}$  is the same as the distance of the original projection  $B_1$  from the ground line  $Ox$ .

The construction of point  $B_4$  when substituting the plane of projections  $\Pi_1$  by  $\Pi_4$  is also shown in Fig. 199, *b* where plane  $\Pi_4$ ,

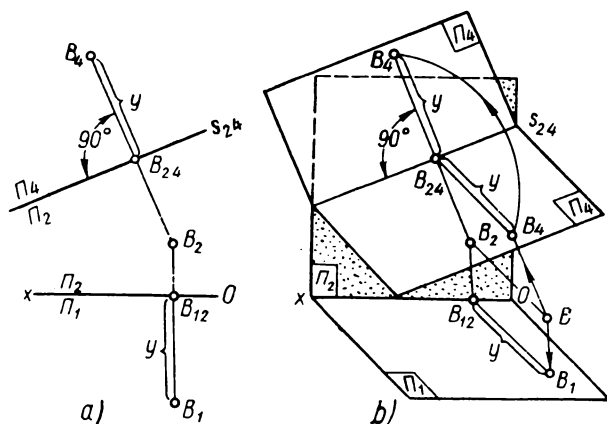


Fig. 199

which is also perpendicular to plane  $\Pi_2$ , is rotated about line  $s_{24}$  together with projection  $B_4$  until it coincides with plane  $\Pi_2$ .

3. The projection of a line, when one of the planes of projection is changed, is received by projecting two of its points onto the new plane of projection and joining up their projections.

Given the projections of a line  $AB$ , see Fig. 200, the sequence in which its new projection on a plane  $\Pi_4$ , perpendicular to the horizontal plane of projection  $\Pi_1$  and parallel to  $AB$ , may be found, is as follows. Since plane  $\Pi_4$  has substituted plane  $\Pi_2$ , the distances of points  $A_4$  and  $B_4$  from line  $s_{14}$  are respectively equal to the distances of points  $A_2$  and  $B_2$  from ground line  $Ox$ .

The new projection  $A_4B_4$  of line  $AB$  is a true length of the line, so the length of  $A_4B_4$  is its real length. It should be recalled that since the plane  $\Pi_4$  is parallel to  $AB$  the ground line  $s_{14}$  is drawn parallel to the horizontal projection  $A_1B_1$  of segment  $AB$ .

Line  $AB$ , as seen in Fig. 201, is projected onto the new plane  $\Pi_4$ , drawn parallel to  $AB$  itself and perpendicular to plane  $\Pi_2$ . The distances from points  $A_4$  and  $B_4$  to line  $s_{24}$  are respectively equal to the distances from points  $A_1$  and  $B_1$  to line  $Ox$ . The new projection  $A_4B_4$  gives the true dimension of line  $AB$  since this line is parallel to  $\Pi_4$ .

The angles  $\alpha$  and  $\beta$ , shown in Figs. 200 and 201, are the true angles of inclination of line  $AB$  to the planes of projection  $\Pi_1$  (angle  $\alpha$ ) and  $\Pi_2$  (angle  $\beta$ ).

Hint. It should be noted that the angle of inclination  $\alpha$  of the given line to the horizontal plane of projection  $\Pi_1$  is given by the angle between the new axis of projection  $s_{14}$ , or of any line parallel to it, and the true length of the line in the system  $\Pi_1-\Pi_4$  which includes the horizontal projection of line  $A_1B_1$ .

The true angle of inclination  $\beta$  of the given line to frontal plane of projection  $\Pi_2$  is seen in the new system  $\Pi_1-\Pi_4$  which includes the original frontal projection of the line.

Finally, it should be understood that when determining the true angle of inclination  $\gamma$  of the given line to the profile plane of projection, it is necessary to construct a new system of planes of projection which must include the profile plane of projection  $\Pi_3$ .

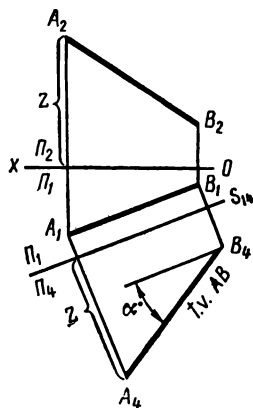


Fig. 200

4. It must be remembered that, although the new planes of projection parallel to line  $AB$  in the transformations analyzed above (see Figs. 200 and 201) may lie at any distance from  $AB$ , the ground line of the new system of planes of projection must necessarily in all cases be parallel to that projection of the line which remains unchanged. In that case the new projection will give the true length of the line.

In Fig. 202, when determining the true length of  $AB$  by the

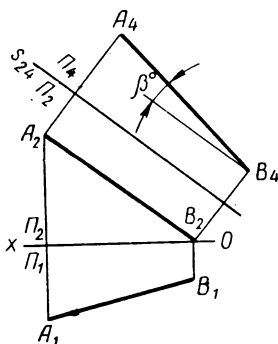


Fig. 201

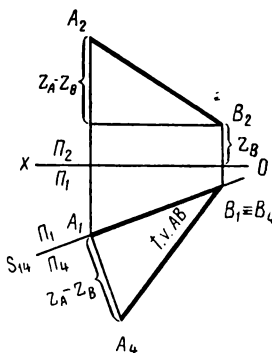


Fig. 202

method of planes of projection, in order to keep the drawing as small as possible, the ground line  $s_{14}$  of the  $\Pi_1$ - $\Pi_4$  system of planes of projection is drawn through the projection  $A_1B_1$ . The size of the drawing may also be limited by laying off from line  $s_{14}$  not the real values of  $z_A$  and  $z_B$  for points  $A$  and  $B$ , but these dimensions less some value. In the case in point  $z_B$  is the value subtracted, and for point  $B$  we receive  $z_B - z_B = 0$ .

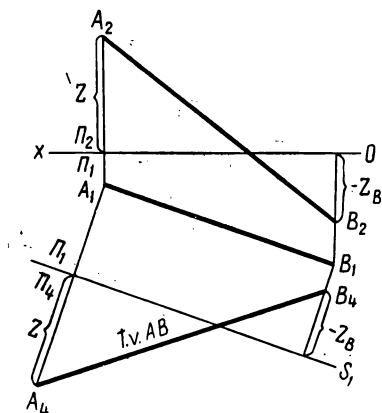


Fig. 203

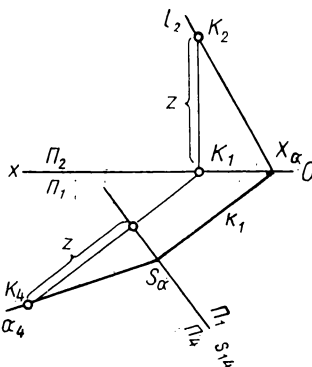


Fig. 204

In this case point  $B_4$  coincides with point  $B_1$  and projection  $A_4B_4$  will be the hypotenuse of the right triangle  $A_1A_4B_4$ , one side of which is the horizontal projection  $A_1B_1$  of the given line and the second side is the difference between distances at which its limiting points  $A$  and  $B$  are situated from plane  $\Pi_1$ , in which the side  $A_1B_1$  of the triangle lies. Thus the right triangle method is actually the conversion of the multiview drawing by the method of changing planes of projection.

5. When constructing a new projection of a line situated in different quadrants the positive coordinates of points should be laid off on one side of the new ground line and the negative coordinates on the other.

A line  $AB$  which contains point  $B$  in the fourth quadrant is given in the drawing (Fig. 203). When constructing a new projection of this line in the  $\Pi_1$ - $\Pi_4$  system of planes, since plane  $\Pi_4$  is perpendicular to plane  $\Pi_1$ , the negative value of  $z_B$  for point  $B$  is laid off upwards from the ground line  $s_{14}$ , whereas the positive coordinate  $z_A$  of point  $A$  is laid off downwards from it.

The projection  $A_4B_4$  gives the true value of the line  $AB$ , since line  $s_{14}$  is drawn parallel to the unchanged projection  $A_1B_1$ , so plane  $\Pi_4$  is also located in space parallel to line  $AB$ .

6. The transformation of an oblique plane  $\alpha$  into a projecting plane by the method of substituting a plane of projection is shown in Fig. 204.

The ground line  $s_{14}$  of the new system of planes of projection  $\Pi_1-\Pi_4$  is drawn perpendicular to trace  $k_1$  of plane  $\alpha$  (see § 16). The point of intersection of trace  $k_1$  with line  $s_{14}$  is a new point of convergence of traces  $S_\alpha$  of plane  $\alpha$ .

In order to find the trace-projection  $\alpha_4$ , which possesses collecting properties since the planes  $\alpha$  and  $\Pi_4$  are mutually perpendicular, on trace  $l_2$  any convenient point  $K$  is taken and projected onto plane  $\Pi_4$ . Then points  $S_\alpha$  and  $K_4$  are joined by a straight line. The lines  $k_1$  and  $\alpha_4$  are traces of plane  $\alpha$  in the system of planes of projection  $\Pi_1-\Pi_4$ .

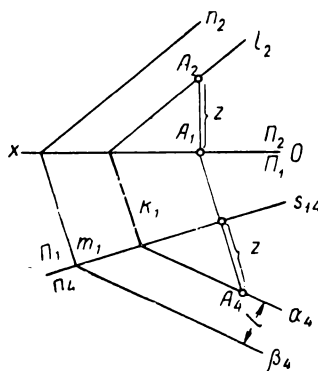


Fig. 205

If the oblique plane  $\alpha$  were given not by traces, but by some other means, for example, by a triangle, then its transformation into a projecting plane could be carried out with the help of horizontals or frontals of plane  $\alpha$ . The ground line of the new system of planes of projection in this case is drawn perpendicular either to the horizontal projection of the horizontal or to the frontal projection of the frontal.

7. The transformation of an oblique plane into a projecting plane simplifies the solution of many problems. Some examples are given below.

Example 1. It is required to determine the distance between the parallel planes  $\alpha$  and  $\beta$  (Fig. 205).

Let us introduce a new plane of projection  $\Pi_4$  perpendicular to planes  $\Pi_1$ ,  $\alpha$  and  $\beta$ . The ground line  $s_{14}$  of the  $\Pi_1-\Pi_4$  system in the multiview drawing should be perpendicular to the traces  $k_1$  and  $m_1$ . Having constructed the trace  $\alpha_4$  of plane  $\alpha$  with the help of point  $A$ , draw the trace  $\beta_4$  parallel to it. The distance between the trace projections  $\alpha_4$  and  $\beta_4$  is the true distance  $l$  between the planes  $\alpha$  and  $\beta$ .

Example 2. It is required to determine the distance of point  $F$  from the plane of a triangle  $ABC$  (Fig. 206).

Let us transform the oblique plane given by triangle  $ABC$  into a projecting plane by the method of substituting a plane of projection (Fig. 206. a). In order to do this in plane  $ABC$  construct the horizontal  $CD$  and draw the new plane of projection  $\Pi_4$  perpendicular to it. The ground line  $s_{14}$  will be perpendicular to the projection  $C_1D_1$  of the horizontal  $CD$ . Having constructed the

projections  $A_4$ ,  $B_4$  and  $C_4$  of the vertices of the triangle, we see that these points lie on straight line  $A_4B_4C_4$ , which is the trace-projection of the plane of triangle  $ABC$  on  $\Pi_4$ , i. e., the trace of a plane which has collecting properties.

Having found the projection  $F_4$  of point  $F$  on the plane  $\Pi_4$ , from it draw a perpendicular to the trace-projection  $A_4B_4C_4$  of plane

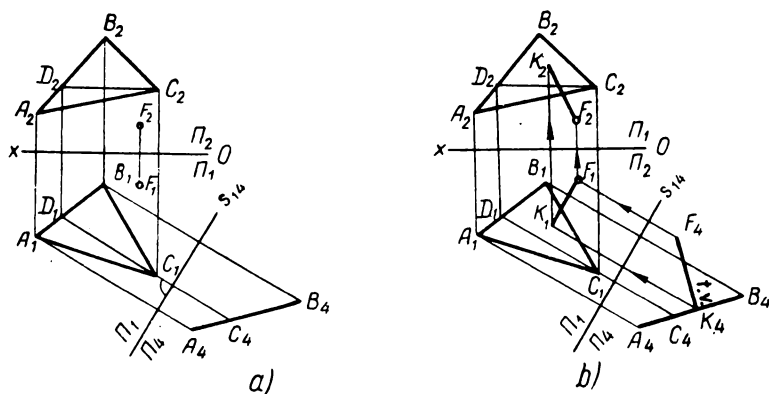


Fig. 206

$ABC$  to obtain point  $K_4$  (Fig. 206, b). The length of  $F_4K_4$  is the true distance of point  $F$  from plane  $ABC$ .

When determining the projection of line  $FK$  on plane  $\Pi_1$  and  $\Pi_2$  it should be remembered that its horizontal projection  $F_1K_1$  must be parallel to ground line  $s_{14}$  since line  $FK$  is parallel to plane  $\Pi_4$ . In other words,  $F_1K_1$  must be perpendicular to projection  $C_1D_1$  of the horizontal of the plane of triangle  $ABC$ , since  $FK$  is perpendicular to plane  $ABC$  (see § 21).

The frontal projection  $K_2$  of point  $K$  is found on the perpendicular projector drawn from  $K_1$ . Point  $K_2$  lies from  $Ox$  at a distance equal to the distance of point  $K_4$  from line  $s_{14}$ .

When a plane is not given by a triangle but by its traces, the procedure remains unchanged. The ground line of the new system of planes of projection is drawn at right angles to one of the traces of the plane.

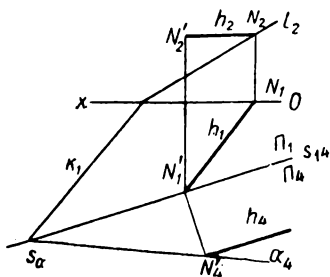


Fig. 207

8. The transformation of an oblique plane, which in the new system of planes  $\Pi_1$ - $\Pi_4$  continues to be an oblique plane, is seen in Fig. 207. The ground line  $s_{14}$  of the new system is inclined to trace  $k_1$ . The point of convergence of the traces of plane  $\alpha$  in the system of planes  $\Pi_1$ - $\Pi_4$  is point  $S_\alpha$ . This is the point of intersec-

tion of the horizontal trace  $k_1$  with the ground line  $s_{14}$ . To find the second point through which trace  $a_4$  of plane  $\alpha$  will pass, in the plane  $\alpha$  draw a convenient horizontal  $h$ . Construct its projection  $h_4$  on plane  $\Pi_4$  ( $N'_4 N'_1 = N_1 N_2$ ). This gives trace  $N'_4$  of this horizontal in plane  $\Pi_4$ . The trace of plane  $\alpha$  in plane  $\Pi_4$  is the straight line joining points  $S_\alpha$  and  $N'_4$ .

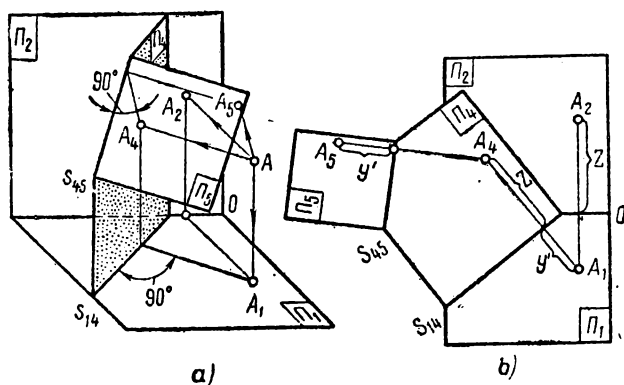


Fig. 208

9. The transformation of projections by substituting two or more planes of projection requires constructions similar to those used when substituting one plane of projection.

The substitution of the principal planes of projection  $\Pi_1$  and  $\Pi_2$  by the additional planes  $\Pi_4$  and  $\Pi_5$  is shown in Fig. 208. The plane  $\Pi_4$  is perpendicular to plane  $\Pi_1$ , whereas plane  $\Pi_5$  is perpendicular to  $\Pi_4$ . For the first transformation plane  $\Pi_2$  is substituted by plane  $\Pi_4$ . Thus a new projection  $A_4$  of the point  $A$  is obtained. The distance of projection  $A_4$  from ground line  $s_{14}$  is equal to the distance of the substituted projection  $A_2$  from ground line  $Ox$ , i. e., equal to the value  $z$  for point  $A$ . In the second conversion plane  $\Pi_1$  is substituted by plane  $\Pi_5$ . In plane  $\Pi_5$  we obtain the second new projection  $A_5$  of point  $A$ . The distance of projection  $A_5$  from the ground line  $s_{45}$  of the second system of planes of projection  $\Pi_4$ - $\Pi_5$  is equal to the distance  $y'$  of the substituted horizontal projection  $A_1$  from the ground line  $s_{14}$  of the  $\Pi_1$ - $\Pi_4$  system of planes of projection.

The multiview drawing is then constructed as follows. Draw line  $s_{14}$  of the first new system of planes of projection  $\Pi_1$ - $\Pi_4$ . Then, having drawn a perpendicular from  $A_1$  to this line, lay off on it from line  $s_{14}$  the value  $z$  for point  $A$ . Point  $A_4$  is then the projection of point  $A$  on plane  $\Pi_4$ . After that draw line  $s_{45}$  of the second new system of planes of projection  $\Pi_4$ - $\Pi_5$  and, on the perpendicular drawn from  $A_4$  to line  $s_{45}$ , lay off from this new ground line distance



$y'$ , so finding point  $A_5$ . The points  $A_4$  and  $A_5$  are the new projections of point  $A$  on planes  $\Pi_4$  and  $\Pi_5$ .

It should be noted that, when the second substitution of a plane of projection was carried out, the dimension  $y'$ , i. e., the distance of projection  $A_5$  to line  $s_{45}$  was taken as equal to the distance from projection  $A_1$  to line  $s_{14}$  of the intermediate system of planes of projection  $\Pi_1-\Pi_4$ . The distance  $y'$  is not equal to the distance of projection  $A_1$  from line  $Ox$  of the principal system of planes of projection  $\Pi_1-\Pi_2$ . The distance  $y'$  in plane  $\Pi_5$  is the length of the fourth line of the four lines situated between the ground lines and projections  $A_5$ ,  $A_4$  and  $A_1$ .

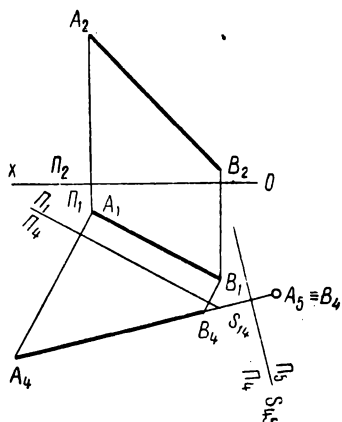


Fig. 209

These lines are the projections of four projecting lines in the main and new systems of planes of projection.

When constructing other projection systems by the method of substituting planes of projection, each new projection of a point will occupy the fourth place, counting from the projection being substituted. For example, in Fig. 218 point  $D_4$  which substitutes the horizontal projection  $D_1$  in the system  $\Pi_2-\Pi_4$ ,  $D_4$  lies at the end of the fourth line, counting 1) from point  $D_1$  to ground line  $Ox$ , 2) from this line to point  $D_2$ , 3) from point  $D_2$  to axis  $s_{24}$ , and finally 4) from axis  $s_{24}$  to point  $D_4$ .

**10.** An oblique line  $AB$  is given in the system of the principal planes of projection  $\Pi_1-\Pi_2$  (Fig. 209). It is required to find a new system of planes of projection in which the given line is projected as a point.

A line is projected as a point when it is perpendicular to the plane of projection. Therefore the new plane of projection must be perpendicular to the given line  $AB$ . It is not possible in one operation to choose a plane of projection at the same time perpendicular both to line  $AB$  and to one of the original planes of projection, since  $AB$  is an oblique line. An intermediate plane of projection parallel to  $AB$  has to be introduced as the first step. The line  $AB$  is then projected onto this plane.

The second plane of projection to be introduced must be perpendicular to both line  $AB$  and the intermediate plane of projection.

In Fig. 209 the intermediate plane of projection  $\Pi_4$  is perpendicular to plane  $\Pi_1$  and parallel to line  $AB$ . The line  $AB$  is projected onto  $\Pi_4$  in its true length, i. e., the length of projection

$A_4B_4=AB$ . The second plane of projection  $\Pi_5$  introduced is perpendicular to  $AB$  and to plane  $\Pi_4$ . So, on the drawing the ground line  $s_{45}$  is perpendicular to projection  $A_4B_4$ . By extending line  $A_4B_4$  to cut line  $s_{45}$  and laying off from line  $s_{45}$  a distance equal to the distance at which both points  $A_1$  and  $B_1$  lie from the intermediate ground line  $s_{14}$ , we obtain a point  $A_5 \equiv B_5$  which is the projection of line  $AB$  on plane  $\Pi_5$ .

This construction is widely used in transforming projections by the method of substituting planes of projection.

11. Examples of the constructions used in descriptive geometry for determining true dimensions by substituting planes of projection are given below.

Example 1. Find the true size of triangle  $ABC$  (Fig. 210). First project triangle  $ABC$  onto plane  $\Pi_4$  perpendicular to plane  $\Pi_1$  and to the plane of the triangle. To do this in the plane of triangle  $ABC$  draw the horizontal  $KC$ . The ground line  $s_{14}$  of the intermediate system of planes must be taken as perpendicular to the projection  $K_1C_1$  of the horizontal. The triangle will be projected onto plane  $\Pi_4$  as line  $A_4C_4B_4$ .

Let us now introduce the second new plane of projection  $\Pi_5$  parallel to the plane of triangle  $ABC$  and perpendicular to plane  $\Pi_4$ . Line  $s_{45}$  must be drawn parallel to the projection  $A_4C_4B_4$ . Having measured the distances at which the horizontal projections  $A_1$ ,  $B_1$  and  $C_1$  of the vertices of the triangle lie from line  $s_{14}$ , lay off these distances on the extensions of the perpendiculars to ground line  $s_{45}$  drawn respectively through points  $A_4$ ,  $B_4$  and  $C_4$ . Having found points  $A_5$ ,  $B_5$  and  $C_5$  join them by straight lines. The triangle  $A_5B_5C_5$  constructed is identical with the original triangle  $ABC$ . In other words, a true view is obtained.

Example 2. Find the shortest distance between the non-intersecting lines  $AB$  and  $CD$  (Fig. 211.).

Here a double substitution of planes of projection is used. The first new plane  $\Pi_4$  is drawn parallel to one of the lines, say, to  $AB$ . Then the line  $AB$  is projected onto this plane in its true length ( $A_4B_4=AB$ ). When line  $CD$  is projected onto plane  $\Pi_4$  its length is

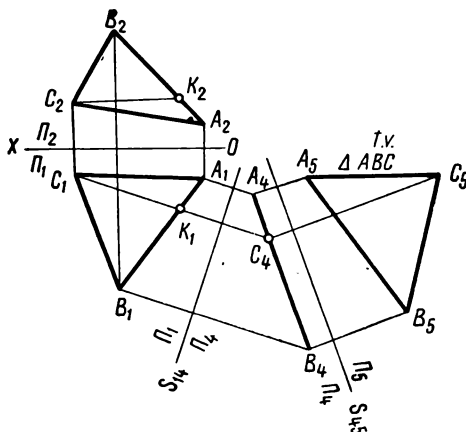


Fig. 210

distorted, since in the  $\Pi_1-\Pi_4$  system of planes of projection it remains an oblique line.

The projections  $A_4, B_4, C_4$  and  $D_4$  of points  $A, B, C$  and  $D$  on plane  $\Pi_4$  are obtained by taking the values of  $z$  for these points from the principal system of planes of projection  $\Pi_1-\Pi_2$ .

The second new plane of projection  $\Pi_5$  is drawn perpendicular to plane  $\Pi_4$  and to line  $AB$ . Then  $AB$  will be projected onto plane

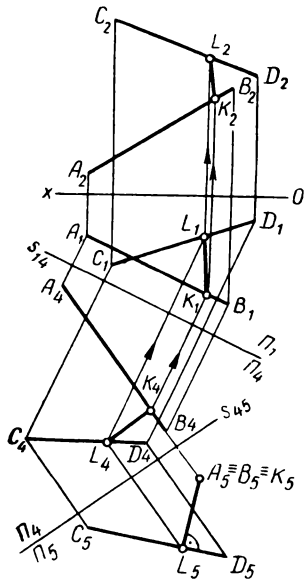


Fig. 211

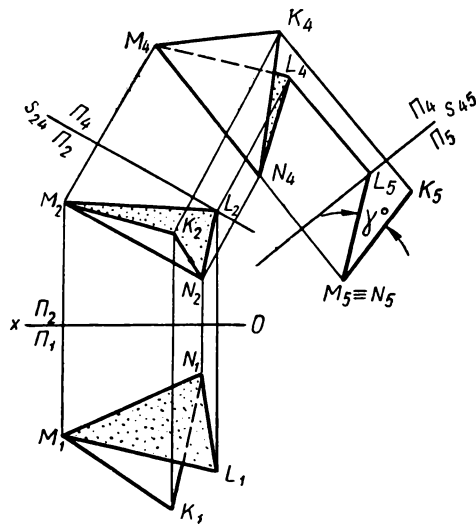


Fig. 212

$\Pi_5$  as a point  $A_5 \equiv B_5$ , and the projection of line  $CD$  onto plane  $\Pi_5$  will be line  $C_5D_5$ .

The projection on plane  $\Pi_5$  of the perpendicular to the lines  $AB$  and  $CD$ , by which the shortest distance between these lines is determined, will give its true length, since this line is parallel to the plane  $\Pi_5$  and line  $AB$  is perpendicular to  $\Pi_5$ . It follows that the right angle between this perpendicular and line  $CD$  when projecting  $CD$  onto plane  $\Pi_4$  will also be seen as a right angle (see § 13).

Therefore having drawn from point  $A_5 \equiv B_5$  a perpendicular to  $C_5D_5$  we find the segment  $K_5L_5$ , giving the shortest distance between lines  $AB$  and  $CD$ . To determine the projection of line  $KL$  in the system of the  $\Pi_1-\Pi_2$  planes of projection first find point  $L_4$ . Then find point  $L_1$ , and finally point  $L_2$  by means of the respective projectors. The point  $K_4$  will be found if a perpendicular is drawn from  $L_4$  to  $A_4B_4$ . The right angle between  $K_4L_4$  and  $A_4B_4$  will not be distorted when projecting onto  $\Pi_4$  since line  $AB$  is parallel to  $\Pi_4$ .

The projections  $K_1$  and  $K_2$  of point  $K$  are also found with the help of projectors.

**Example 3.** It is required to determine the true value of the angle formed between the planes of the triangles  $KMN$  and  $LMN$  (Fig. 212).

The true size of the dihedral angle is given by the angle formed by the straight line traces of the planes in a plane perpendicular to the edge or line of intersection of these given planes or by the orthographic projection of the dihedral angle onto a plane perpendicular to their line of intersection (Fig. 213).

In the case, shown in Fig. 212, the line  $MN$  of intersection of the given planes is an oblique line. So it is necessary to transform the projection of the angle so that the line of intersection is perpendicular to the new plane of projection.

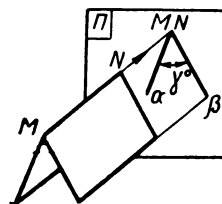


Fig. 213

First, project the triangles onto plane  $\Pi_4$  which is perpendicular to plane  $\Pi_2$  and parallel to the line of intersection  $MN$ . The ground line  $s_{24}$  is drawn parallel to the projection  $M_2N_2$  of the line of intersection which remains unchanged. To obtain the projection of the triangles  $KMN$  and  $LMN$  on plane  $\Pi_4$  from points  $M_2, N_2, K_2$  and  $L_2$  draw perpendiculars to line  $s_{24}$ . On the extension of these perpendiculars lay off from  $s_{24}$  the values of  $y$  for points  $M, N, K$  and  $L$ . These distances are the distances of the projections of  $M_1, N_1, K_1$  and  $L_1$  from ground line  $Ox$ . The line of intersection  $MN$  of the given planes then projects onto  $\Pi_4$  in its true value ( $M_4N_4 = MN$ ).

Next, the new plane of projection  $\Pi_5$ , perpendicular to plane  $\Pi_4$  and to  $MN$ , is introduced. The ground line  $s_{45}$  of the system of planes, of projection  $\Pi_4-\Pi_5$  is drawn perpendicular to projection  $M_4N_4$ . The edge  $MN$  is projected onto plane  $\Pi_5$  as a point  $M_5 \equiv N_5$ . When constructing projections  $K_5$  and  $L_5$  of points  $K$  and  $L$  it is borne in mind that the distances of points  $M_5, N_5, K_5$  and  $L_5$  from line  $s_{45}$  are equal to the distances of points  $M_2, N_2, K_2$  and  $L_2$  from ground line  $s_{24}$ , respectively. Points  $K_5$  and  $L_5$  are joined to point  $M_5 \equiv N_5$ . The plane angle  $K_5M_5L_5$  will be the true dihedral angle formed by the given planes  $KMN$  and  $LMN$ .

When planes forming a dihedral are given by their traces the procedure for substituting projections in order to obtain the plane dihedral angle remains the same.

**Example 4.** It is required to determine the angle between line  $AB$  and plane  $\alpha$  given by the triangle  $CDE$  (Fig. 214).

If the line and plane were situated relative to the principal planes of projection  $\Pi_1$  and  $\Pi_2$  in such a way that the line was parallel to  $\Pi_2$  and the plane parallel to  $\Pi_1$ , then the angle  $\gamma$

between them would be projected without distortion onto one of the principal planes of projection, in the case shown in Fig. 215, onto  $\Pi_2$ .

In the example shown in Fig. 214, the line  $AB$  and the given plane are both oblique. The drawing must be transformed so that, in the new system of planes of projection, the line is parallel to one

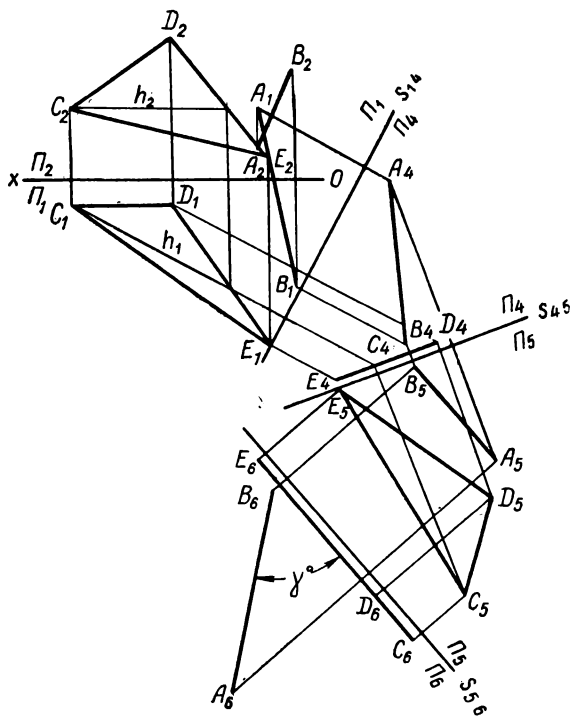


Fig. 214

plane of projection and the plane is parallel to the other plane of projection of the new system.

The first new plane  $\Pi_4$  is drawn perpendicular to both plane  $\Pi_1$  and plane  $\alpha$ . The triangle  $CDE$  will then be projected onto plane  $\Pi_4$  as line  $C_4D_4E_4$ , and the line  $AB$  as line  $A_4B_4$ .

The second new plane of projection,  $\Pi_5$ , should be perpendicular to  $\Pi_4$  and parallel to plane  $\alpha$ . Then the plane of triangle  $CDE$  will be projected onto  $\Pi_5$  as a true value ( $C_5D_5E_5 = CDE$ ), and the line  $AB$  as  $A_5B_5$ .

Finally, the third new plane  $\Pi_6$  is drawn perpendicular to plane  $\Pi_5$  and parallel to line  $AB$ . The projection of line  $AB$  on plane  $\Pi_6$  will be a true value ( $A_6B_6 = AB$ ). The projection of triangle  $CDE$

will be line  $C_6E_6D_6$  parallel to the ground line  $s_{56}$  of the  $\Pi_5-\Pi_6$  system of planes of projection.

As a result of three successive substitutions of planes of projection, i. e.,  $\Pi_2$  by  $\Pi_4$ ,  $\Pi_1$  by  $\Pi_5$ ,  $\Pi_4$  by  $\Pi_6$  we obtain the  $\Pi_5-\Pi_6$  system in which the projections of the line and plane are those shown in Fig. 215. The angle  $\gamma$  contained between lines  $A_6B_6$  and  $C_6E_6D_6$  is the true size of the angle formed by line  $AB$  and the plane of triangle  $CDE$  (Fig. 214).

Example 5. It is required to construct the projection of a sphere the surface of which must contain points  $A, B, D, F$ , not situated in one plane (Fig. 216).

It can be seen that the given points  $A, B, D, F$  in reality are not contained in one plane. The auxiliary line  $DK$ , drawn so that its frontal projection  $D_2K_2$  passes through projection  $F_2$  of point  $F$  in space. However, this line does not pass through point  $F$ , since  $F_1$  does not lie on  $D_1K_1$ . It follows that point  $F$  is not contained in the plane of triangle  $ABD$ .

The centre of the sphere passing through the four given points  $A, B, D$  and  $F$  may be found as follows (Fig. 217):

- 1) construct in plane  $ABD$  a circle with its centre in point  $M$ , passing through points  $A, B$  and  $D$ ;
- 2) in plane  $ABF$  construct a circle with its centre in point  $N$ , passing through point  $A, B$  and  $F$ ;
- 3) through points  $M$  and  $N$  draw lines respectively perpendicular to planes  $ABD$  and  $ABF$ . The intersection of these perpendiculars will determine the centre  $C$  of the sphere and the distance from point  $C$  to any of the given points  $A, B, D, F$  will be the radius of the sphere.

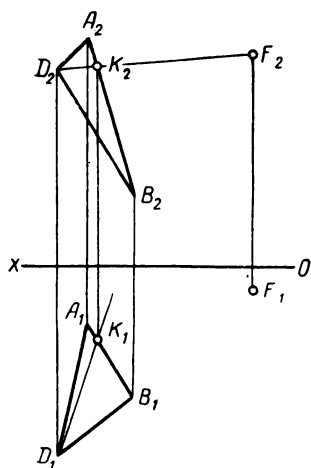


Fig. 216

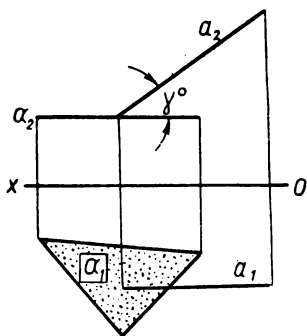


Fig. 215

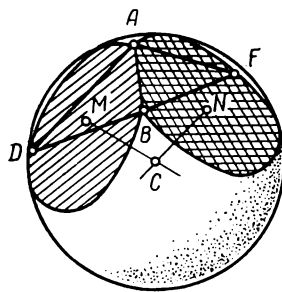


Fig. 217

Let us now construct the multiview drawing (Fig. 218). First, the  $\Pi_1$ - $\Pi_2$  system of planes of projection is substituted by the  $\Pi_2$ - $\Pi_4$  system in which the new plane of projection  $\Pi_4$  is parallel to the line  $AB$  in which the planes  $ABD$  and  $ABF$  intersect, i. e., parallel to the edge of the dihedral angle. The ground line  $s_{24}$  is therefore drawn parallel to the projection  $A_2B_2$  of this edge. The

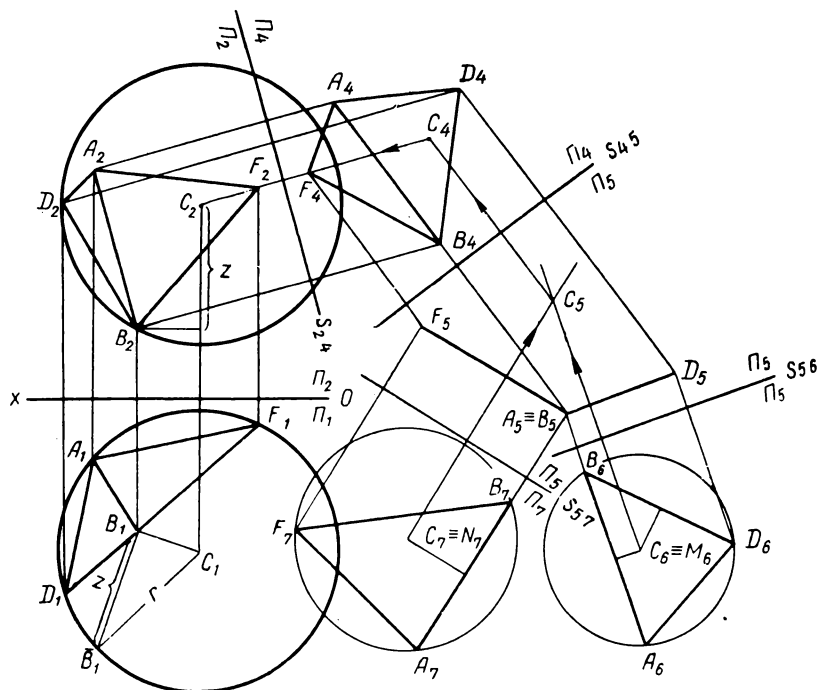


Fig. 218

projection of  $AB$  on plane  $\Pi_4$  is a true value ( $A_4B_4=AB$ ). Having found  $D_4$  and  $F_4$  we next find the projection of the dihedral planes on  $\Pi_4$ .

Further, we project the planes forming the angle onto plane  $\Pi_5$  in the  $\Pi_4$ - $\Pi_5$  system. The plane  $\Pi_5$  is drawn perpendicular to  $AB$ , i. e., the ground line  $s_{45}$  is perpendicular to  $A_4B_4$ . The edge  $AB$  is projected onto plane  $\Pi_5$  as point  $A_5 \equiv B_5$ . The projection of triangle  $ABD$  is line  $A_5D_5$ , and that of triangle  $ABF$  line  $A_5F_5$ .

Angle  $D_5A_5F_5$  gives the true dihedral angle of planes  $ABD$  and  $ABF$ .

The other two new planes of projection,  $\Pi_6$  and  $\Pi_7$ , are to be perpendicular to plane  $\Pi_5$ . They are also to be parallel to the planes of triangles  $ABD$  and  $ABF$ , respectively. So ground line  $s_{56}$  is drawn

parallel to  $A_5D_5$  and ground line  $s_{57}$  parallel to  $A_5F_5$ . The triangle  $ABD$  is projected onto plane  $\Pi_6$  in true size ( $A_6B_6D_6=ABD$ ), and triangle  $ABF$  is also projected onto plane  $\Pi_7$  in its true size ( $A_7B_7F_7=ABF$ ).

Next, determine the centres  $M_6$  and  $N_7$  of the circles passing, respectively, through the vertices of the two triangles in these projections. To do this draw perpendiculars to the midpoints of the sides of the triangles. These circles so constructed are the projections of sections of the required sphere when it is cut by the planes of triangles  $ABD$  and  $ABF$ .

The projection  $C_5$  of the centre  $C$  of the sphere on plane  $\Pi_5$  is found by drawing a perpendicular to ground line  $s_{56}$  through point  $M_6$  and a perpendicular to ground line  $s_{57}$  through point  $N_7$  to meet in  $C_5$ .

By means of reverse projections find  $C_4$ ,  $C_2$  and  $C_1$  which are the projections of point  $C$  on planes  $\Pi_4$ ,  $\Pi_2$  and  $\Pi_1$ . Point  $C_4$  lies at a distance from line  $s_{45}$  equal to the distance of projection  $M_6$  from line  $s_{56}$ , which is the same as the distance of  $N_7$  from  $s_{57}$ . It should be recalled that points  $M_6$  and  $N_7$  coincide with projections  $C_6$  and  $C_7$  of centre  $C$ .

Having found the principal projections  $C_1$  and  $C_2$  of the centre of the sphere join them by a straight line with the corresponding projections on  $\Pi_1$  and  $\Pi_2$  of any of the four given points, for example, with point  $B$ , and, by constructing a right triangle  $B_1C_1\bar{B}_1$ , determine the true length  $CB$ , which is equal to the radius  $r$  of the sphere ( $C_1\bar{B}_1=CB$ ).

With a radius  $r$  describe circles from points  $C_1$  and  $C_2$ . These are the horizontal and frontal projections of the sphere, the surface of which passes through the given points  $A$ ,  $B$ ,  $D$  and  $F$ .

**Example 6.** In Fig. 219 the transformation of the principal projections of a detail is carried out by successively substituting the planes of projection  $\Pi_2$  and  $\Pi_1$  by new planes, i. e., by  $\Pi_4$  perpendicular to  $\Pi_1$ , and  $\Pi_5$  perpendicular to  $\Pi_4$ . With the help of this construction the transition from an orthographic multiview drawing to the axonometric representation of the detail and its cut-out on plane  $\Pi_5$  is achieved.

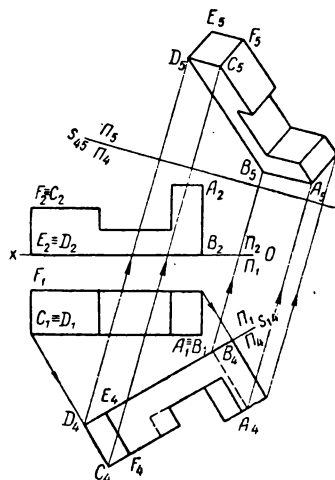


Fig. 219



## Problems and Exercises

1. Does a change in the distance of a new plane of projection from the projected object have any bearing on the final projection?
  2. How should a new plane of projection be placed so that an oblique line is projected onto it in its true length?
  3. In what position must a triangle be situated if its true size can be determined by substituting only one plane of projection?
  4. What transformations must be carried out in order to project an oblique line onto a new plane of projection as a point?
  5. How must a new plane of projection be placed in order that a plane figure is projected onto it as a straight line?
  6. When can the true value of a dihedral angle be determined without distortion by substituting only one of the planes of projection?
  7. When can the shortest distance between two non-intersecting lines be determined by introducing only one new plane of projection?
  8. Determine the true length of an oblique line  $AB$  and its angles of inclination to planes  $\Pi_1$  and  $\Pi_2$  (Fig. 220).
  9. Draw a perpendicular from a point  $O$  to the plane of triangle  $ABC$  (Fig. 221).
  10. Determine the distance between the parallel planes  $\alpha$  and  $\beta$  (Fig. 222).
  11. Determine the distance from point  $E$  to the plane given by parallel lines  $AB$  and  $CD$  (Fig. 223).
  12. Determine the true dimension of the quadrangle  $ABCD$  (Fig. 224).
  13. Determine angles  $\alpha$  and  $\beta$  formed by plane  $\alpha$  with planes of projection  $\Pi_1$  and  $\Pi_2$  (Fig. 222).
- Hint. Angle  $\alpha$  may be found in the system  $\Pi_1-\Pi_4$  if the new ground line  $s_{14}$  is drawn perpendicular to the traces  $k_1$  and  $m_1$ . Angle  $\beta$  is found in the  $\Pi_2-\Pi_5$  system with ground line  $s_{24}$  perpendicular to traces  $l_2$  and  $n_2$ .
14. From point  $K$  draw a perpendicular to the line of intersection of the plane of triangle  $ABC$  with plane  $\alpha$  given by traces  $k$  and  $l$  (Fig. 225).
  15. Determine the angle formed by the plane of triangle  $ABC$  with the plane  $\alpha$  (Fig. 225).
  16. Draw a plane at the same distance from both the non-intersecting lines  $AB$  and  $CD$  (Fig. 226).

Hint. The construction is carried out as follows. Through point  $B$  draw a line parallel to  $CD$ . Then draw a level line in the plane of parallelism obtained. This level line enables us to find the new system of planes of projection in which the projection of the plane of parallelism is a straight line. The required plane will be constructed in this same system of planes of projection.

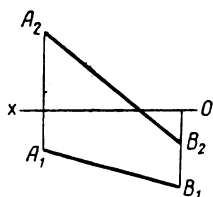


Fig. 220

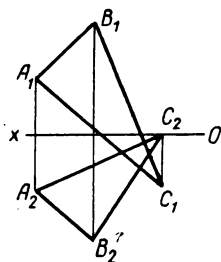


Fig. 221

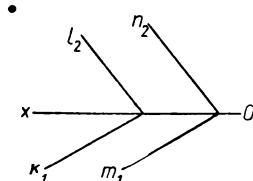


Fig. 222

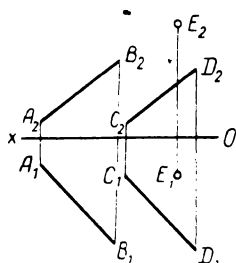


Fig. 223

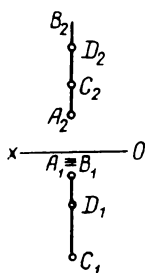


Fig. 224

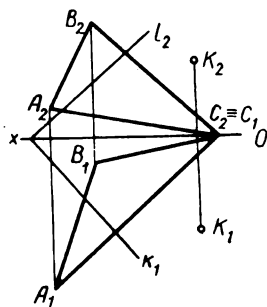


Fig. 225

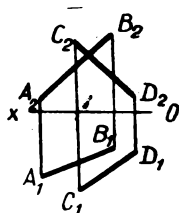


Fig. 226

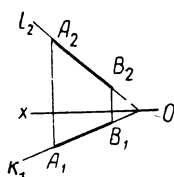


Fig. 227

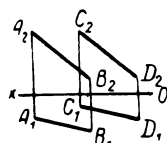


Fig. 228

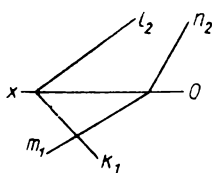


Fig. 229

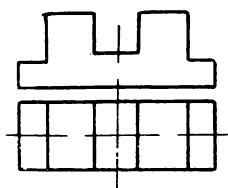


Fig. 230

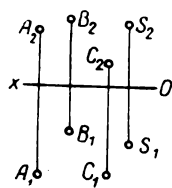


Fig. 231

17. Determine the angle between line  $AB$  and plane  $\alpha$  given by the traces  $k$  and  $l$  (Fig. 227).

Hint. This problem may be solved as in example 4, § 24, (see Fig. 214).

18. Determine the distance between parallel lines  $AB$  and  $CD$  (Fig. 228).

Hint. Both line segments should first be projected onto a new plane in their true length. Then, by drawing a second supplementary plane perpendicular to the lines received, project them as points. The distance between these points is the required distance.

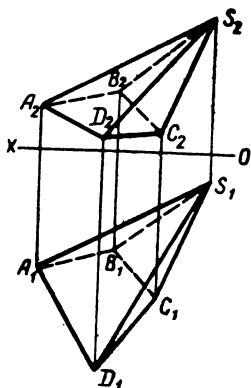


Fig. 232

19. Construct a bisecting plane between planes  $\alpha$  and  $\beta$  (Fig. 229).

Hint. This problem can be solved by changing two planes of projection (see Fig. 212 and related text). First, the line of intersection of the given planes should be determined.

20. Construct new projections of the three-dimensional detail (Fig. 230) on planes  $\Pi_4 \perp \Pi_2$  and  $\Pi_5 \perp \Pi_4$ .

21. Four points not lying in one plane are shown in Fig. 231. Construct a cone with one of the points as the vertex. The other three points should lie on the lateral surface of the cone.

22. Given a pyramid  $SABCD$  (Fig. 232). Determine: a) the angle between the base  $ABCD$  and face  $SAB$ , b) the altitude of the pyramid, c) the shortest distance between edges  $AD$  and  $CB$ .

23. Draw a circle through the vertices of any triangle  $ABC$ , the plane of which is oblique.

24. On ground line  $Ox$  find a point equidistant from traces  $M$  and  $N$  of a given line  $AB$ . (See Figs. 209, 211 and 212 and related text.)

## § 25. Method of Revolution

In accordance with this method the principal system of planes of projection  $\Pi_1-\Pi_2$  remains unchanged while the lines and plane figures to be projected are rotated about corresponding axes to positions in which their projections in the multiview drawings become the true views of the lines, figures, etc.

The axes of revolution are straight lines perpendicular to a plane of projection, parallel to it, or contained in a plane of projection.

Oblique lines are generally not used as axes of rotation since they complicate the construction.

If it becomes necessary in the process of construction to revolve

any geometrical element about an oblique axis, the axis is first brought to some suitable position, after which the necessary rotation is carried out.

## § 26. Revolution about Axes Perpendicular to Planes of Projection

1. Let there be given point  $A$  and the axis of revolution  $i$  perpendicular to the horizontal plane of projection  $\Pi_1$  (Fig. 233, *a*). When rotated about axis  $i$  point  $A$  will describe a circle in a plane

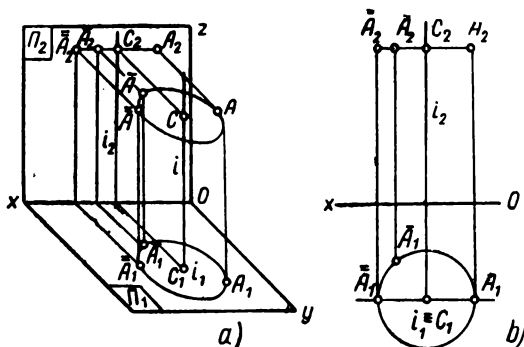


Fig. 233

perpendicular to axis  $i$ . The radius of this circle is equal to the distance of point  $A$  from axis  $i$ . The centre of the circle of revolution is therefore contained in the axis of rotation  $i$ . The points  $\bar{A}$ ,  $\bar{A}$  are the various positions of point  $A$  on the circumference as it is rotated.

Since the plane of the circle of revolution of point  $A$  is parallel to plane  $\Pi_1$ , the circle projects on this plane without distortion, but on plane  $\Pi_2$  it will project as a line, the length of which is equal to the diameter of the circle. Moreover, this line is parallel to ground line  $Ox$ .

In the multiview drawing (Fig. 233, *b*) the following elements are given: projections  $i_1$  and  $i_2$  of the axis of revolution  $i$ , the projection of the circle of revolution of point  $A$  and projections  $C_1$  and  $C_2$  of centre  $C$  of this circle. The projection  $C_1$  coincides with point  $i_1$  which is the horizontal projection of the axis of revolution  $i$ . If point  $A$  revolving about axis  $i$  has been moved into position  $\bar{A}$ , its horizontal projection occupies the position  $\bar{A}_1$  on the horizontal projection of the circle of rotation. Its frontal projection is  $\bar{A}_2$  on the frontal projection of the circle. The points  $\bar{A}_1$  and  $\bar{A}_2$  lie on the same perpendicular to the ground line  $Ox$ .

When the axis of revolution  $i$  is perpendicular to the frontal plane of projection  $\Pi_2$  (Fig. 234), the circle of revolution of point

$A$  is contained in a frontal plane parallel to  $\Pi_2$  and so is projected onto plane  $\Pi_2$  as a circle and as a line parallel to ground line  $Ox$  onto plane  $\Pi_1$ .

It follows that a circle of revolution of a point is projected without distortion onto that plane of projection to which the axis of revolution is perpendicular. On the other principal plane of projection the circle is projected as a line parallel to the ground line  $Ox$ .

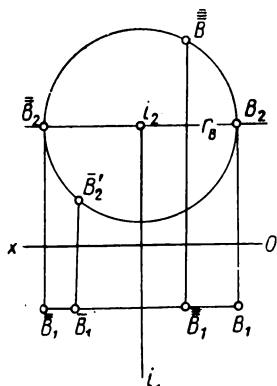


Fig. 234

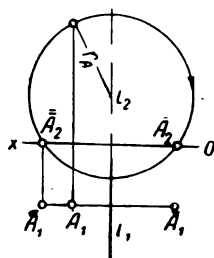


Fig. 235

2. Let us consider some examples.

Example 1. Revolve point  $A$  about the axis  $i$  perpendicular to plane  $\Pi_2$  until it coincides with plane  $\Pi_1$  (Fig. 235).

The circle of revolution of the point in this case is contained in the frontal plane, therefore it will be projected onto  $\Pi_2$  without distortion and onto  $\Pi_1$  as a line parallel to ground line  $Ox$ .

Join projection  $i_2$  of the axis of revolution  $i$ , with which the frontal projection of the centre of the circle of revolution coincides, with point  $A_2$ , and with point  $i_2$  as the centre describe a circle having its radius  $r_A = A_2i_2$ . This circle cuts ground line  $Ox$  in points  $\bar{A}_2$  and  $\bar{A}_2$ .

These points are the frontal projections of the two possible positions  $\bar{A}$  and  $\bar{A}$  of point  $A$  when it coincides with plane  $\Pi_1$ . The horizontal projections  $\bar{A}_1$  and  $\bar{A}_1$  of points  $\bar{A}$  and  $\bar{A}$  are obtained by drawing projectors from  $\bar{A}_2$  and  $\bar{A}_2$  to intersection with line  $\bar{A}_1\bar{A}_1$ , which is the projection of the circle of revolution on plane  $\Pi_1$  and parallel to ground line  $Ox$ .

Example 2. By rotation about the axis  $i$  perpendicular to plane  $\Pi_1$ , make point  $A$  coincide with plane  $\alpha$  (Fig. 236).

Through point  $A$  pass the horizontal plane  $\beta$  in which point  $A$  moves when revolved about axis  $i$ . Plane  $\beta$  will cut the given plane  $\alpha$  along the horizontal  $NK$ .

Point  $A$  will coincide with plane  $\alpha$  when the arc of the circle,

which this point describes, cuts the horizontal  $NK$ . So, to obtain the horizontal projection  $\bar{A}_1$  of point  $A$  in its position  $\bar{A}$  in plane  $\alpha$  it is necessary to describe an arc  $A_1\bar{A}_1$  from the centre  $i_1$  with a radius  $r_A$  to cut the horizontal projection  $N_1K_1$  of the horizontal  $NK$ . The frontal projection  $\bar{A}_2$  of the new position of point  $A$  in plane  $\alpha$  is at the intersection of the projector passing through  $A_1$  with the frontal projection  $N_2K_2$  of the horizontal  $NK$ . This problem, like the previous one, has two solutions.

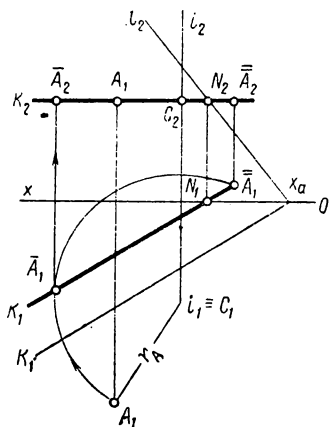


Fig. 236

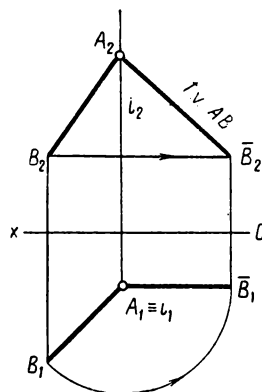


Fig. 237

3. The revolution of a straight line about an axis intersecting the line is reduced to the revolution of one of its points, because a point of intersection of the line with the axis of revolution remains stationary and, together with the point, which is moved, will determine the new position of the line.

In the drawing (Fig. 237) an oblique line  $AB$  is revolved about the axis  $i$  passing through point  $A$  perpendicular to plane  $\Pi_1$  until it becomes parallel to the plane of projection  $\Pi_2$ . Then the projection  $A_2\bar{B}_2$  on  $\Pi_2$  gives the true length of the line  $AB$ .

Point  $A$  remains stationary. Its horizontal projection  $A_1$  coincides with the horizontal projection of line  $i$  ( $A_1 \equiv i_1$ ). The projection  $A_1B_1$  is equal in length to the radius of revolution of point  $B$ .

If line  $AB$  is to be parallel to plane  $\Pi_2$ , its horizontal projection  $A_1B_1$  must be rotated until it is parallel to ground line  $Ox$ . This is done by moving point  $B_1$  through arc  $B_1\bar{B}_1$  to a new position  $\bar{B}_1$ . The frontal projection  $B_2$  moves along line  $B_2\bar{B}_2$  parallel to ground line  $Ox$ . Having drawn a projector through  $\bar{B}_1$  point  $\bar{B}_2$  is found at the intersection of the projector and line  $B_2\bar{B}_2$ . Point  $\bar{B}_2$  is the frontal projection of point  $B$  in its new position  $\bar{B}$ . By joining the

points  $A_2$  and  $\bar{B}_2$  we obtain the frontal projection  $A_2\bar{B}_2$  of the given line after rotation about axis  $i$ . The length of  $A_2\bar{B}_2$  is the true length of  $AB$ .

4. If the axis of revolution does not intersect a given straight line two points of the line are rotated through the same angle and in the same direction.

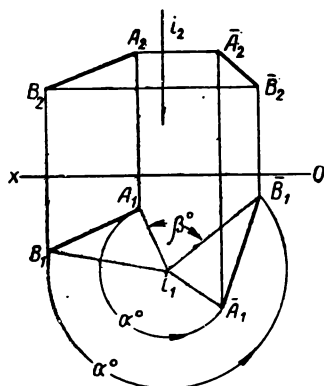


Fig. 238

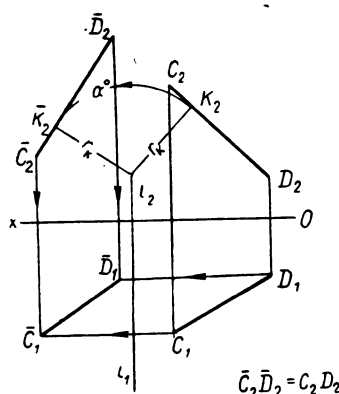


Fig. 239

Let line  $AB$  be rotated about axis  $i$  through an angle  $\alpha$  in a counter-clockwise direction (Fig. 238). Having moved point  $A_1$  through the arc  $A_1\bar{A}_1$  with radius  $A_1i_1$  to point  $\bar{A}_1$ , and point  $B_1$  through the arc  $B_1\bar{B}_1$  with radius  $B_1i_1$  to point  $\bar{B}_1$ , both moving through one and the same angle  $\alpha$ , we find the horizontal projection  $\bar{A}_1\bar{B}_1$  of line  $AB$  after moving through angle  $\alpha$ . The frontal projection  $A_2\bar{B}_2$  is obtained by passing projectors through points  $\bar{A}_1$  and  $\bar{B}_1$ . These projectors intersect with the lines drawn through points  $A_2$  and  $B_2$  parallel to ground line  $Ox$ .

In this construction the triangles  $A_1B_1i_1$  and  $\bar{A}_1\bar{B}_1i_1$  are identical, since two of their sides and the angle between them remain unchanged.

That is to say,  $A_1i_1 = \bar{A}_1i_1$ ,  $B_1i_1 = \bar{B}_1i_1$ ,  $\angle A_1i_1B_1 = \angle \bar{A}_1i_1\bar{B}_1$ , since each of these angles equals  $360^\circ - (\alpha^\circ + \beta^\circ)$ .

Since these triangles  $A_1B_1i_1$  and  $\bar{A}_1\bar{B}_1i_1$  are equal, their third sides are equal, i. e.,  $A_1B_1 = \bar{A}_1\bar{B}_1$ .

It follows that the projection of any line or figure on the plane, to which the axis of rotation is perpendicular, after rotation remains unchanged.

This deduction simplifies the construction of projections of given lines revolved about an axis perpendicular to the plane of projection and not intersecting the given line.

Such a simplified construction is shown in Fig. 239. The line  $CD$ , when revolved counter-clockwise through an angle  $\alpha$  about axis  $i$  which is perpendicular to plane  $\Pi_2$ , takes the new position of the line  $\overline{CD}$ . From point  $i_2$  a perpendicular to the projection  $C_2D_2$  is drawn. The base  $K_2$  of this perpendicular is moved through an angle  $\alpha$  to point  $\overline{K}_2$ . Point  $\overline{K}_2$  is joined to point  $i_2$  by a straight line. Then through point  $\overline{K}_2$  a line at an angle  $90^\circ$  to  $i_2\overline{K}_2$  is drawn. From point  $\overline{K}_2$  on either side distances  $\overline{K}_2\overline{C}_2$  and  $\overline{K}_2\overline{D}_2$  are laid off respectively equal to the lengths  $K_2C_2$  and  $K_2D_2$ . The line  $\overline{C}_2\overline{D}_2$  then is the projection of  $CD$  on plane  $\Pi_2$  in its new position  $\overline{CD}$ .

The horizontal projections  $\overline{C}_1$  and  $\overline{D}_1$  of points  $\overline{C}$  and  $\overline{D}$  are found with the help of projectors drawn through points  $\overline{C}_2$  and  $\overline{D}_2$  to intersect lines drawn parallel to ground line  $Ox$  through  $\overline{C}_1$  and  $\overline{D}_1$ . By joining up points  $\overline{C}_1$  and  $\overline{D}_1$  the horizontal projection of  $CD$  in its new position  $\overline{CD}$  is obtained.

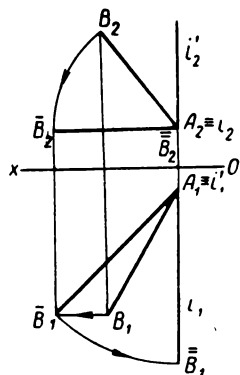


Fig. 240

5. Revolution about two axes. Fig. 240 illustrates the method of bringing an oblique line  $AB$  into a position, perpendicular to plane  $\Pi_2$ , by successive revolutions of this segment about two axes respectively perpendicular to the planes of projection  $\Pi_1$  and  $\Pi_2$ .

First, we revolve line  $AB$  about axis  $i$  perpendicular to plane  $\Pi_2$ ; it also passes through point  $A$ . Line  $AB$  is brought into position  $\overline{AB}$  parallel to plane  $\Pi_1$ . The frontal projection  $A_2\overline{B}_2$  will be parallel to ground line  $Ox$  and the horizontal projection  $A_1\overline{B}_1$  gives the true length of  $AB$ .

The second axis of rotation  $i'$  is drawn through point  $A$  perpendicular to plane  $\Pi_1$ . The line is rotated about this axis until it becomes perpendicular to plane  $\Pi_2$ . The horizontal projection  $A_1\overline{B}_1$  will then be perpendicular to ground line  $Ox$  and the frontal projection  $A_2\overline{B}_2$  becomes a point.

6. Let us consider some examples.

Example 1. It is required to construct a straight line  $AB$  inclined at angles  $\alpha$  and  $\beta$  to planes  $\Pi_1$  and  $\Pi_2$ .

If line  $KL$ , making an angle  $\alpha$  with plane  $\Pi_1$  (Fig. 241), revolves about an axis  $i$ , perpendicular to plane  $\Pi_1$  and passing through point  $K$ , point  $L$  will move along a circle lying in plane  $\Pi_1$  and the line  $KL$  in each new position ( $\overline{KL}$ ;  $\overline{K\overline{L}}$ ;  $\overline{K\overline{L}}$ ;  $\overline{K\overline{L}}$  and so on) will form



one and the same angle  $\alpha$  with plane  $\Pi_1$ . The surface formed by rotation of the line is a conical surface.

If point  $K$  remains stationary and line  $KL$  is made to form an angle  $\beta$  with plane  $\Pi_2$ , after which it is revolved about the axis passing through point  $K$  and perpendicular to plane  $\Pi_2$ , then point  $L$  will describe another circle, this time in the frontal plane, and the movement of line  $KL$  will describe a new conical surface.

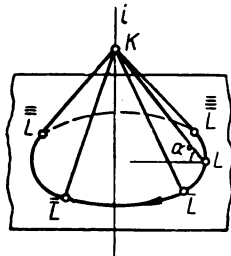


Fig. 241

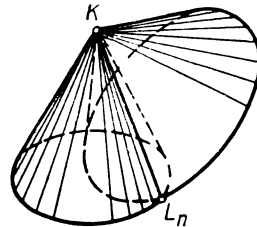


Fig. 242

All points of each of the two circles are at one and the same distance  $KL$  from point  $K$ . Therefore both circles will lie on the surface of a sphere, the centre of which coincides with point  $K$  and the radius of which is equal to the length of line  $KL$ .

Let the circles intersect in some point  $L_n$  (Fig. 242). Then line  $KL_n$  will be common to both the first and second cones and so form angle  $\alpha$  with plane  $\Pi_1$  and angle  $\beta$  with plane  $\Pi_2$ .

These considerations enable the following construction to be carried out in the multiview drawing (Fig. 243).

Draw the projections  $A_1B_1$  and  $A_2B_2$  of the frontal line  $AB$  at an angle  $\alpha$  to plane  $\Pi_1$ . Through point  $A$  pass the axis  $i$  perpendicular to plane  $\Pi_1$ . Revolve the line  $AB$  about this axis. Point  $B$  will describe a circle with radius  $A_1B_1$ , since when rotated about axis  $i$  line  $AB$  forms a conical surface with its vertex at point  $A$ .

Further, leaving point  $A$  stationary, draw the horizontal  $\overline{AB}$  equal in length to  $AB$  and making an angle  $\beta$  with plane  $\Pi_2$ . The horizontal projection  $A_1\overline{B}$  of line  $\overline{AB}$  is equal in length to  $AB$  and makes an angle  $\beta$  with ground line  $Ox$ .

Through point  $A$  draw the second axis of revolution perpendicular to plane  $\Pi_2$ . Revolve line  $\overline{AB}$  about this axis. The point  $\overline{B}$  in the frontal plane, parallel to  $\Pi_2$ , will describe a circle with radius  $A_2\overline{B}_2$ . The movement of line  $\overline{AB}$  generates the second conical surface with its vertex at the same point  $A$ .

The circles of radii  $A_1B_1$  and  $A_2\overline{B}_2$  will intersect in points  $\overline{\overline{B}}$  and  $\overline{\overline{B}}$ . Joining up these points with point  $A$  by lines  $A\overline{\overline{B}}$  and  $A\overline{\overline{B}}$  we obtain the two positions of line  $AB$  forming angles  $\alpha$  and  $\beta$  with planes  $\Pi_1$  and  $\Pi_2$  as was required.

The lines, symmetrical to the lines constructed,  $A\bar{B}$  and  $A\bar{\bar{B}}$ , with respect to the frontal plane passing through point  $A$ , will also form angles  $\alpha$  and  $\beta$  with planes  $\Pi_1$  and  $\Pi_2$ . These lines may be obtained if both nappes of one of the cones are constructed.

It follows that, as a rule, this problem has four solutions. If the circles do not intersect but touch, the problem has only one solution. The required line will then be a profile line. In general the

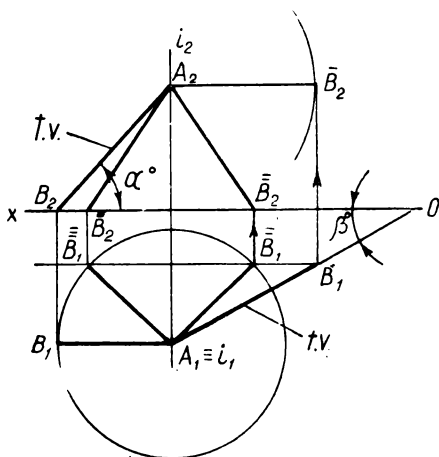


Fig. 243

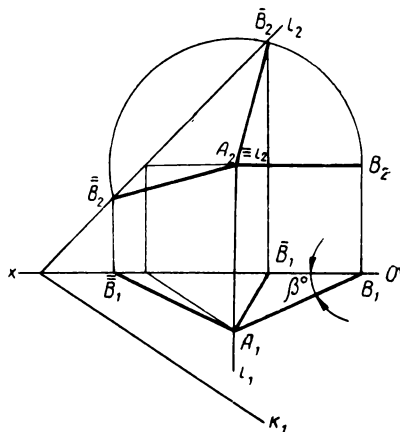


Fig. 244

problem may be solved only in the case when the sum of the angles  $\alpha$  and  $\beta$  is less than  $90^\circ$ , which corresponds to the possible positions of a line in space in relation to planes  $\Pi_1$  and  $\Pi_2$ .

**Example 2.** It is required to draw through point  $A$  a line in plane  $\alpha$  at an angle  $\alpha$  to plane  $\Pi_2$  (Fig. 244).

The problem is solved by constructing a conical surface with its vertex in point  $A$  and the generatrix  $AB$  in its first position parallel to plane  $\Pi_1$  and forming an angle  $\alpha$  with plane  $\Pi_2$ , point  $B$  being contained in plane  $\Pi_2$ .

Through point  $A$  draw the axis of revolution  $i$  perpendicular to plane  $\Pi_2$ . Then revolve line  $AB$  about this axis. The point  $B$  will describe a circle in plane  $\Pi_2$  and the line  $AB$  will generate a conical surface. The circle with a radius  $A_2B_2$  intersects the frontal trace  $l_2$  of plane  $\alpha$  in points  $\bar{B}$  and  $\bar{\bar{B}}$ . Joining these points with point  $A$  we find in this case two possible solutions. The lines  $A\bar{B}$  and  $A\bar{\bar{B}}$  are situated in plane  $\alpha$  and make an angle  $\alpha$  with plane  $\Pi_2$ .

If the circle which has been constructed merely touches trace  $l_2$ , the problem has one solution and in that case line  $AB$  and plane  $\alpha$  have one and the same angle of inclination  $\alpha$  to the plane of projection  $\Pi_2$ .

When the circle and the trace  $l_2$  have no common points it becomes impossible to draw in plane  $\alpha$  a line inclined at angle  $\alpha$  to plane  $\Pi_2$ .

**Example 3.** By rotation about axis  $i$  which is perpendicular to plane  $\Pi_1$ , make the given line  $AB$  coincide with plane  $\alpha$  (Fig. 245).

Having found the piercing point  $K$  of line  $AB$  with plane  $\alpha$  through it draw the axis  $i$  perpendicular to plane  $\Pi_1$ . Rotate about this axis one of the extreme points of  $AB$ , for example point  $A$ , until it coincides with plane  $\alpha$ . This operation is carried out

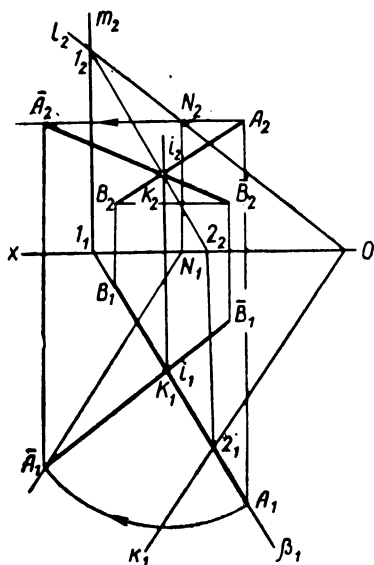


Fig. 245

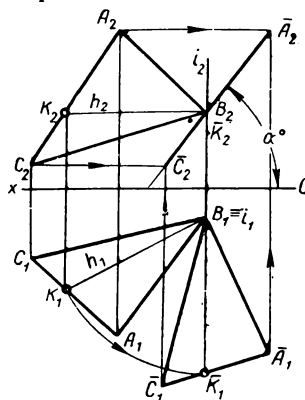


Fig. 246

with the help of the horizontal  $AN$  drawn in plane  $\alpha$  through point  $A$  (see Fig. 236).

Having found the position  $\bar{A}$  of point  $A$ , when it coincides with plane  $\alpha$ , join point  $K$  with point  $\bar{A}$ . Line  $\bar{A}K$  is that part of line  $AB$  which has been rotated to coincide with plane  $\alpha$ .

The position  $\bar{B}$  of the point  $B$  of the line, when it has been made to coincide with  $\alpha$ , may be constructed by extending the projection  $\bar{A}_2K_2$  beyond point  $K_2$  to intersect a line drawn through point  $B_2$  parallel to ground line  $Ox$ . The point  $\bar{B}_2$  is the frontal projection of the point  $\bar{B}$  after rotation. Then with the help of a projector find the point  $\bar{B}_1$  on the extension of projection  $\bar{A}_1K_1$ .

7. The revolution of a plane about an axis perpendicular to the plane of projection is performed by rotating the points and lines by which the plane is given through one and the same angle and in the same direction.

It is required to revolve the plane of triangle  $ABC$  into a position perpendicular to plane of projection  $\Pi_2$  (Fig. 246).

All horizontals of a horizontal-projecting plane are projected onto plane  $\Pi_2$  as points since these lines in space are perpendicular to plane  $\Pi_2$  (see Fig. 110). Therefore in the triangle  $ABC$  draw any horizontal, say  $BK$ , and revolve it about the axis  $i$  perpendicular to plane  $\Pi_1$  and passing through point  $B$  until it takes up the position  $B\bar{K}$  and is perpendicular to plane  $\Pi_2$ , namely  $B_1\bar{K}_1 \perp Ox$ , and the projection  $B_2\bar{K}_2$  becomes a point. The horizontal projection of the triangle, when rotated through the same angle as the projection  $B_1K_1$  of the horizontal was moved, will occupy the position  $\bar{A}_1B_1\bar{C}_1$ . The triangle  $\bar{A}_1B_1\bar{C}_1$  is equal to triangle  $A_1B_1C_1$ .

To obtain the frontal projections  $\bar{A}_2$  and  $\bar{C}_2$  of points  $\bar{A}$  and  $\bar{C}$  through  $A_2$  and  $C_2$  draw lines parallel to  $Ox$ , and intersect them by projectors passing through points  $\bar{A}_1$  and  $\bar{C}_1$ . By joining points  $\bar{A}_2$ ,  $B_2$  and  $\bar{C}_2$  line  $\bar{A}_2B_2\bar{C}_2$  is obtained. This is the frontal trace-projection of the given triangle plane when turned until it is perpendicular to the frontal plane of projection  $\Pi_2$ .

The angle  $\alpha$  formed by the trace-projection  $\bar{A}_2B_2\bar{C}_2$  and ground line  $Ox$  determines the true value of inclination of triangle  $ABC$  to the horizontal plane of projection  $\Pi_1$ .

8. When the plane is given by traces it can be rotated by revolving one of its traces and a point or line belonging to the plane.

In Fig. 247 the plane  $\alpha$  assigned by traces  $k_1$  and  $l_2$  is turned about axis  $i$  perpendicular to plane  $\Pi_1$  clockwise through angle  $\beta$ . The revolution of the horizontal trace into position  $\bar{k}_1$  is carried out with the help of the perpendicular  $i_1K_1$  drawn from  $i_1$  to  $k_1$ . After moving point  $K_1$  into the position  $\bar{K}_1$  and joining it up with point  $i_1$  of line  $i_1\bar{K}_1$  draw through point  $\bar{K}_1$  the perpendicular  $\bar{k}_1$ . This line, which is the horizontal trace of plane  $\alpha$  after rotation, intersects ground line  $Ox$  in point  $X_{\alpha}^-$ , the point of convergence of traces of plane  $\alpha$  in its new position  $\bar{\alpha}$ .

The axis of rotation  $i$ , as can be seen from the drawing, is contained in plane  $\Pi_2$  and intersects trace  $l_2$  in point  $A$ . This point  $A$  is located on the axis of rotation  $i$  and so remains stationary. By joining the frontal projection  $A_2$  of point  $A$  with point  $X_{\alpha}^-$  we find the new frontal trace  $\bar{l}_2$  of plane  $\alpha$  after it has been revolved about axis  $i$  through angle  $\beta$ .

In Fig. 248 the axis  $i$  about which plane  $\alpha$  is revolved into the position  $\bar{\alpha}$ , is perpendicular to plane  $\Pi_1$  but is not contained in plane  $\Pi_2$ . The position of the new frontal trace  $\bar{l}_2$  is determined by



therefore the angle  $\beta$  between trace  $\bar{k}_1$  and ground line  $Ox$  is the required angle of inclination of plane  $\alpha$  to  $\Pi_2$ .

**Example 2.** It is required to determine the distance between the parallel oblique planes  $\alpha$  and  $\beta$  (Fig. 250).

Convert both planes into frontal-projecting planes by revolving them about the axis  $i$  which is perpendicular to plane  $\Pi_1$  and lying in plane  $\Pi_2$  (see example 1). The planes  $\alpha$  and  $\beta$  after the revolution are perpendicular to plane  $\Pi_2$ . The distance  $h$  between their new frontal traces  $\bar{l}_2$  and  $\bar{n}_2$  is the required distance.

**Example 3.** It is required to revolve plane  $\alpha$  about the axis  $i$  perpendicular to plane  $\Pi_1$  so that it passes through point  $A$  (Fig. 251).

In plane  $\alpha$ , at the altitude of point  $A$ , draw the horizontal  $NL$  and revolve it about axis  $i$  until it passes through point  $A$ . Point  $L_1$  which is the foot of the perpendicular drawn from  $i_1$  to  $N_1L_1$  will describe a circle when the horizontal is rotated. By drawing a tangent to this circle through point  $A_1$  the new position  $\bar{N}_1\bar{L}_1$  of the horizontal projection of the horizontal  $NL$  of plane  $\alpha$  is found.

Point  $K_1$ , which is the foot of the perpendicular drawn from  $i_1$  to trace  $k_1$  when plane  $\alpha$  is revolved, will also describe a circle. A tangent drawn to this circle parallel to the projection  $\bar{N}_1\bar{L}_1$  of the horizontal  $\bar{NL}$  gives the horizontal trace  $\bar{k}_1$  of plane  $\alpha$  in its new position  $\bar{\alpha}$ .

The frontal trace  $\bar{l}_2$  will pass through  $X_{\bar{\alpha}}$ , i. e., the point of convergence of the traces, and through the frontal trace  $\bar{N}_2$  of the horizontal  $\bar{NL}$ .

This problem has two solutions, because through point  $A_1$  two tangents to the circle with radius  $i_1L_1$  can be drawn.

**Example 4.** It is required to find the true size of a triangle  $ABC$  contained in an oblique plane (Fig. 252).

This problem is solved by successive revolutions of triangle  $ABC$  about two axes respectively perpendicular to the planes of projection  $\Pi_1$  and  $\Pi_2$ .

By means of the first revolution about axis  $i$  perpendicular to plane  $\Pi_2$ , passing through vertex  $A$ , triangle  $ABC$  is brought into a position  $\bar{A}\bar{B}\bar{C}$  so that it becomes perpendicular to plane  $\Pi_1$  (see Fig. 246). The projection of axis  $i$  is not shown in Fig. 252. The

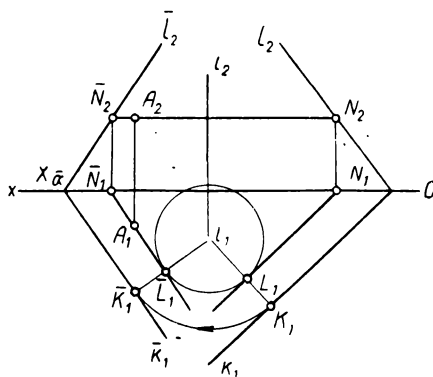


Fig. 251

frontal  $AK$  is drawn in the plane of the triangle and then rotated about axis  $i$  until it becomes perpendicular to plane  $\Pi_1$ .

The triangle  $\overline{ABC}$  is brought into position  $\overline{\overline{ABC}}$  parallel to plane  $\Pi_2$  by revolving it about axis  $i'$  perpendicular to plane  $\Pi_1$ . This axis passes through point  $\overline{C}$ . The projection of axis  $i'$  is not shown in the drawing. The horizontal projection  $\overline{\overline{AB_1C_1}}$  is parallel to ground line  $Ox$ , and the frontal projection  $\overline{\overline{A_2B_2C_2}}$  is a true view of the given triangle  $ABC$ .

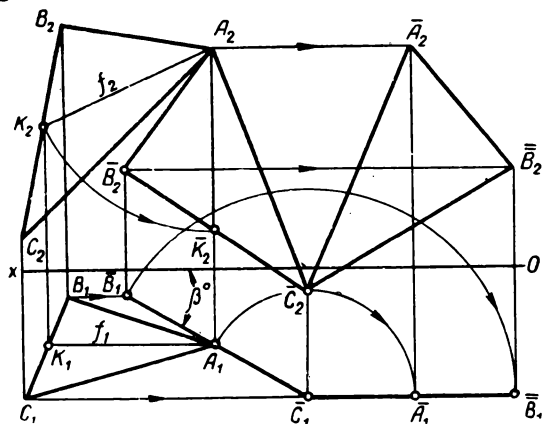


Fig. 252

The first revolution about the axis  $i$  is sufficient to determine the angle of inclination  $\beta$  of triangle  $ABC$  plane to the frontal plane of projection  $\Pi_2$ .

#### Problems and Exercises

1. In what plane does a point move when it is revolved about an axis a) perpendicular to plane  $\Pi_1$ , b) perpendicular to plane  $\Pi_2$ ?

2. To which plane of projection should the axis of revolution be perpendicular in order to revolve an oblique line a) into a horizontal position, b) into a frontal position?

3. What revolutions about axes perpendicular to the planes of projection are required to bring an oblique line into a vertical position?

4. How can an oblique plane, given as a triangle, be rotated so as to be perpendicular a) to plane  $\Pi_1$ , b) to plane  $\Pi_2$ ?

5. The same as (4) when the plane is given by its traces.

6. Revolve a point  $A$  through  $60^\circ$  in a clockwise direction about an axis  $i$  perpendicular to plane  $\Pi_1$  (Fig. 253).

7. Make point  $A$  coincide with the plane of triangle  $BCD$  by rotating it about an axis  $i$  perpendicular to plane  $\Pi_1$  (Fig. 254).

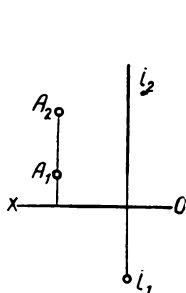


Fig. 253

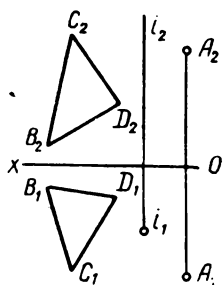


Fig. 254

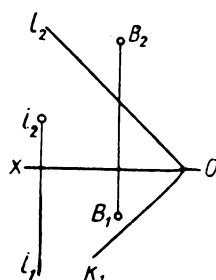


Fig. 255

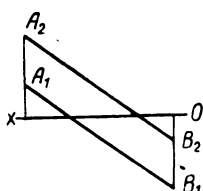


Fig. 256

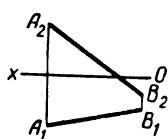


Fig. 257

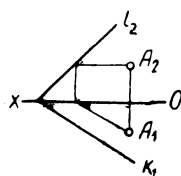


Fig. 258

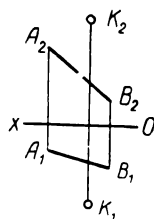


Fig. 259

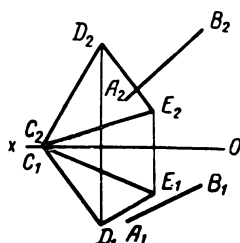


Fig. 260

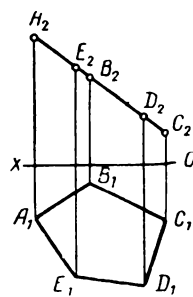


Fig. 261

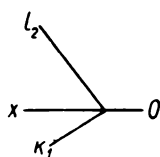


Fig. 262

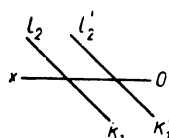


Fig. 263



Hint. In plane  $BCD$  draw an auxiliary horizontal at the altitude of point  $A$ .

8. Bring point  $B$  into a position so that it lies at 10 mm from plane  $\alpha$  (Fig. 255) using the method of rotation about an axis  $i$  perpendicular to  $\Pi_2$ .

Hint. First, a plane  $\beta$  should be drawn parallel to plane  $\alpha$  at a distance of 10 mm from it.

9. Determine the angle  $\beta$  formed by line  $AB$  and plane  $\Pi_2$  (Fig. 256).

10. Rotate line  $AB$  until it becomes perpendicular to plane  $\Pi_1$  (Fig. 257).

Hint. First, line  $AB$  should be converted into a frontal by turning it about an axis perpendicular to plane  $\Pi_1$ . Then, it should be revolved about an axis perpendicular to plane  $\Pi_2$ .

11. Draw a line  $AB$  in plane  $\alpha$  to form an angle of  $45^\circ$  with  $\Pi_1$ . Point  $A$  is given (Fig. 258). (See Fig. 244 and text.)

12. By means of rotation determine the distance from point  $K$  to line  $AB$  (Fig. 259).

Hint. Two successive revolutions of the line and the point about axes perpendicular to the planes of projection should be carried out so as to make the line perpendicular to one of the planes of projection. The line  $AB$  will then be projected as a point. The distance between this point and the new projection of point  $K$  will be the required distance.

13. Revolve line  $AB$  until it lies in the plane of triangle  $CDE$  (Fig. 260). (See Fig. 245 and text.)

14. Revolve line  $AB$  until it is parallel to the plane of triangle  $CDE$  (Fig. 260).

Hint. This problem may be solved in the same way as the previous one. However, the line  $AB$  should be made to coincide with a plane drawn parallel to the given plane  $CDE$ .

15. Revolve the oblique line  $AB$  about an axis  $i$  perpendicular to plane  $\Pi_1$  so that its projection  $A_1B_1$  becomes parallel to the projection  $K_1L_1$  of a line  $KL$  contained in plane  $\Pi_1$ .

16. Construct a line  $AB$  so that the angles it makes with planes  $\Pi_1$  and  $\Pi_2$  are  $\alpha=45^\circ$  and  $\beta=30^\circ$ , respectively. (See Fig. 243 and text.)

17. Determine the true size of the pentagon  $ABCDE$  (Fig. 261).

18. Rotate the plane  $\alpha$  so that it becomes a frontal projecting plane (Fig. 262).

Hint. Revolve the plane about an axis perpendicular to  $\Pi_1$  (compare with Fig. 249).

19. Determine the distance between the parallel planes  $\alpha$  and  $\beta$  (Fig. 263).

20. Revolve a triangle  $ABC$  which is part of an oblique plane so that it becomes parallel to plane  $\Pi_2$ .

## § 27. Revolution about Axes Parallel to Planes of Projection

1. An oblique line  $AB$  is shown in Fig. 264. Let a line  $i$  be drawn parallel to plane  $\Pi_1$  intersecting line  $AB$  in point  $K$ . Assuming line  $i$  to be the axis of revolution let line  $AB$  be rotated until it becomes parallel to plane  $\Pi_1$ .

After the line has assumed its new position  $\bar{A}\bar{B}$  its frontal projection  $\bar{A}_2\bar{B}_2$  will merge with the frontal projection  $i_2$  of the axis of revolution  $i$  and its horizontal projection  $\bar{A}_1\bar{B}_1$  will give the true length of  $AB$ .

The construction of the horizontal projection  $\bar{A}_1\bar{B}_1$  after the line has been turned is carried out as follows. The points  $A$  and  $B$  are revolved about axis  $i$  and they move in horizontal-projecting planes  $\alpha$  and  $\beta$  perpendicular to the axis of revolution  $i$ . It follows, that the projection  $\bar{A}_1$  and  $\bar{B}_1$  of the extremes of line  $AB$  in its new position  $\bar{A}\bar{B}$  will lie on the traces  $\alpha_1$  and  $\beta_1$  perpendicular to the projection  $A_1B_1$ . When line  $\bar{A}\bar{B}$  is in a horizontal position the radii of revolution of points  $A$  and  $B$  will be projected onto plane  $\Pi_1$  in their true length. The true length of the radius  $r_A$  for point  $A$  is found with the help of a right triangle. The radius  $r_A$  is laid off from point  $C_1$ , which is the horizontal projection of the centre of revolution  $C$  for point  $A$ . Point  $\bar{A}_1$  lies on trace  $\alpha_1$ .

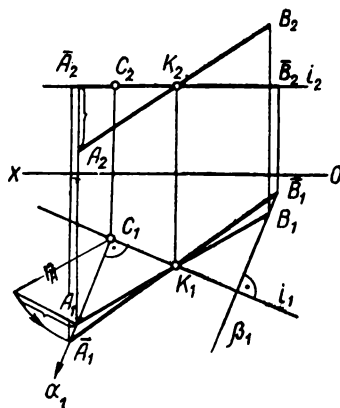


Fig. 264

Point  $\bar{A}_1$  is joined by a line with the projection  $K_1$  of the stationary point  $K$  in which axis  $i$  intersects line  $AB$ . In this way the horizontal projection of line  $AB$  after rotation about axis  $i$  is obtained. The horizontal projection  $\bar{B}_1$  of point  $B$  lies at the intersection of projection  $A_1K_1$  with trace  $\beta_1$ . The projection  $\bar{A}_1\bar{B}_1$  gives the true length of line  $AB$ .

2. The construction of the true view of a plane figure by revolution about an axis parallel to a plane of projection is shown in the multiview drawing (Fig. 265). By rotation about axis  $i$ , which is parallel to plane  $\Pi_1$ , the triangle  $ABC$  is brought to a position parallel to plane  $\Pi_1$ , on which it is then projected in its true size ( $\triangle A_1\bar{B}_1\bar{C}_1 = \triangle ABC$ ).

The frontal projection  $A_2\bar{B}_2\bar{C}_2$  of triangle  $ABC$  after revolution merges with the frontal projection of axis  $i$ .

In order to construct triangle  $A_1\bar{B}_1\bar{C}_1$  a perpendicular to the projection  $i_1$  of the axis of revolution is drawn. Then the true dimension of radius  $r_B$  is determined by turning point  $B$ . To do this the right triangle method is employed. The distance  $r_B$  is marked off on the perpendicular. Point  $\bar{B}_1$  obtained in this way is the projection of the vertex  $\bar{B}$  of the original triangle when it becomes parallel to plane  $\Pi_1$ .

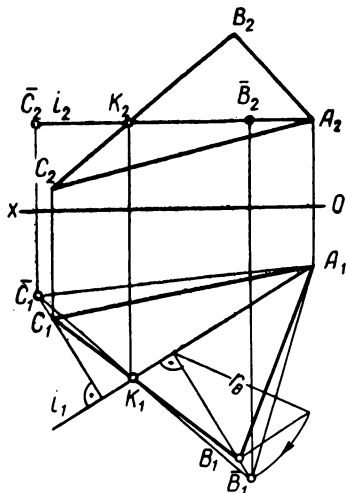


Fig. 265

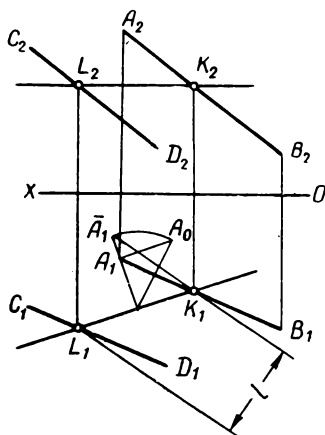


Fig. 266

Point  $\bar{C}_1$ , which is the horizontal projection of vertex  $C$  of the triangle  $ABC$  when it is parallel to plane  $\Pi_1$ , is found by drawing a line through points  $\bar{B}_1$  and  $K_1$  to intersect the perpendicular drawn from  $\bar{C}_1$  to  $i_1$ .

The vertex  $A$  of the triangle remains stationary since it is a point on the axis of revolution. Its projection  $A_1$  is joined with projections  $\bar{B}_1$  and  $\bar{C}_1$  by straight lines. This gives the horizontal projection  $A_1\bar{B}_1\bar{C}_1$  of triangle  $\bar{A}\bar{B}\bar{C}$  when it is parallel to plane  $\Pi_1$ .

**Example.** Determine the distance between the parallel lines  $AB$  and  $CD$  (Fig. 266).

Draw the horizontal  $KL$  of the plane determined by the lines  $AB$  and  $CD$ . About this horizontal lines  $AB$  and  $CD$  should be rotated so that they become parallel to plane  $\Pi_1$ .

Projection  $\bar{A}_1$  of point  $A$  is found by determining the radius  $r_A$ . A straight line is drawn through  $\bar{A}_1$  and  $K_1$ . This is the horizontal projection of line  $AB$  after rotation. Then, through projection  $L_1$  of the stationary point  $L$  a second line is drawn parallel to line  $\bar{A}_1K_1$ . The distance  $l$  between these lines gives the true distance at which  $AB$  lies from  $CD$ .

## § 28. The Method of Coincidence

1. The method of coincidence is a particular case of the method of revolution. It is used to bring geometrical elements, points, lines and planes into coincidence with one of the planes of projection by means of revolution about an axis contained in that plane of projection.

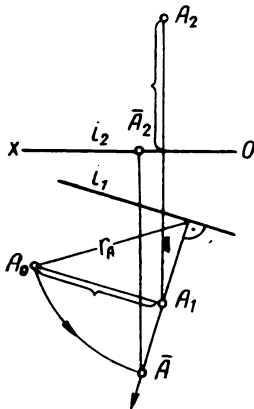


Fig. 267

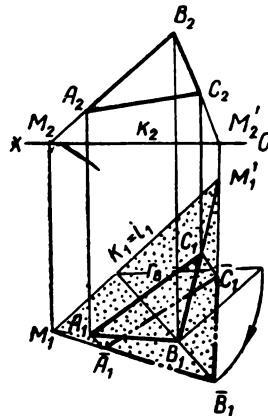


Fig. 268

In the drawing (Fig. 267) it is seen that point  $A$  is brought into coincidence with the horizontal plane of projection  $\Pi_1$  by revolving it about axis  $i$  contained in that plane. Point  $\bar{A}$  is the position of point  $A$  in  $\Pi_1$  and it lies on a perpendicular drawn from  $A_1$  to  $i_1$ . Its distance from axis  $i$  is equal to the radius of revolution  $r_A$  of point  $A$ . The radius  $r_A$  is obtained by the right-triangle method. The frontal projection  $\bar{A}_2$  of  $A$  after it has been made to coincide with  $\Pi_1$  is located on the ground line  $Ox$ .

2. The method of bringing a plane given as the triangle  $ABC$  into coincidence with plane  $\Pi_1$  is shown in Fig. 268. The horizontal trace of this plane serves as the axis of revolution. By rotation about the axis  $k$  triangle  $MBM'$  is made to lie in plane  $\Pi_1$ . When carrying out this construction only the vertex  $B$  has to be brought into coincidence with  $\Pi_1$ , since the vertices  $M$  and  $M'$  are themselves contained in plane  $\Pi_1$ . Having found point  $\bar{B}_1$  join it by straight lines to  $M_1$  and  $M'_1$ . This gives triangle  $M_1\bar{B}_1M'_1$  which coincides with  $\Pi_1$ . Triangle  $\bar{A}\bar{B}\bar{C}$  is part of triangle  $M_1\bar{B}_1M'_1$ . To find this triangle it is sufficient to draw from points  $A_1$  and  $C_1$  lines  $A_1\bar{A}_1$  and  $C_1\bar{C}_1$  perpendicular to the projection  $k_1$  of the axis of rotation. These lines intersect the sides  $M_1\bar{B}_1$  and  $M'_1\bar{B}_1$  of triangle  $M_1\bar{B}_1M'_1$ .



4. In order to bring the frontal trace  $l_2$  of plane  $\alpha$  which is parallel to ground line  $Ox$  into coincidence with plane  $\Pi_1$  (Fig. 271), a point  $A$  is taken on trace  $l_2$  and brought into coincidence with plane  $\Pi_1$  by the right triangle method. In the new position of the plane trace  $\bar{l}_2$  is parallel to trace  $k_1$  and passes through point  $\bar{A}_2$  which lies in plane  $\Pi_1$ .

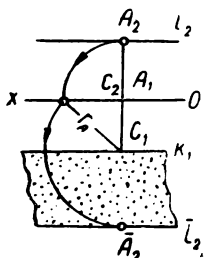


Fig. 271

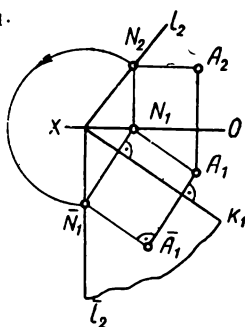


Fig. 272

5. A point contained in a given plane may be brought into coincidence with a plane of projection with the help of an auxiliary line drawn in the given plane through the point. A horizontal or frontal is generally chosen as the auxiliary line.

The construction required to bring point  $A$  contained in plane  $\alpha$  into coincidence with plane  $\Pi_1$  is shown in Fig. 272.

Through point  $A$  in plane  $\alpha$  the horizontal  $AN$  is drawn. The frontal trace  $N \equiv N_2$  of this horizontal is made to coincide with plane  $\Pi_1$ . Through point  $\bar{N}_1$  and point of convergence of traces  $X_\alpha$  the trace  $\bar{l}_2$  of plane  $\alpha$ , when it coincides with  $\Pi_1$ , is drawn. The horizontal  $\bar{N}_1\bar{A}_1$  is drawn through point  $\bar{N}_1$  parallel to  $k_1$ . This line is coincident with  $\Pi_1$ . Point  $\bar{A}_1$  lies on this line at its intersection with the perpendicular drawn from  $A_1$  to  $k_1$  or, what is the same thing, to  $\bar{N}_1\bar{A}_1$ .

This construction could be carried out by using a frontal lying in plane  $\alpha$ , as shown in Fig. 273.

The same number of graphical operations is required to determine point  $\bar{A}_1$  by the methods, shown in Figs. 272 and 273. Nevertheless when bringing several points into coincidence with plane  $\Pi_1$  it is advisable to use a frontal as the auxiliary line, because after the first point has been determined in its coincided position, the rest of the construction requires less graphical work and is simpler than when a horizontal is used (see Fig. 273).

It also follows that, when bringing a number of points of a given plane into coincidence with the frontal plane of projection  $\Pi_2$ , it is better to use a horizontal of the plane as the auxiliary.

6. The method of coincidence is very convenient when constructing a figure of given dimensions and form in an oblique plane. The given plane may be brought into coincidence with one of the planes of projection  $\Pi_1$  or  $\Pi_2$ . The required figure is then constructed as a true view coinciding with the plane of projection. The given plane with the figure on it is returned to its original position. This gives the projections of the figure on  $\Pi_1$  and  $\Pi_2$ . Such an operation is called the *restoration* of points to the plane.

It is required to construct in plane  $\alpha$  a regular hexagon with sides equal in length to line  $h$  (Fig. 274).

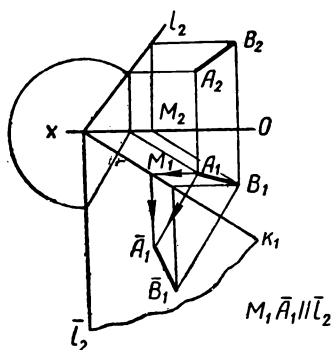


Fig. 273

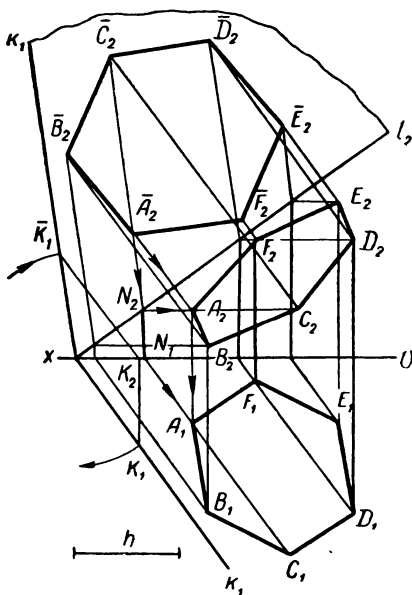


Fig. 274

After making plane  $\alpha$  coincide with  $\Pi_2$  with the help of point  $K$  taken on trace  $k_1$  on plane  $\alpha$  in its new position construct the regular hexagon  $\bar{A}_2\bar{B}_2\bar{C}_2\bar{D}_2\bar{E}_2\bar{F}_2$  with its sides equal in length to  $h$ .

After this, with the help of horizontals in the coincided plane  $\bar{\alpha}$  drawn parallel to the horizontal trace  $\bar{k}_1$  of the plane and passing through the vertices of the hexagon we construct the frontal projection  $A_2B_2C_2D_2E_2F_2$  and the horizontal projection  $A_1B_1C_1D_1E_1F_1$  of the figure. The construction of the frontal and horizontal projections of one vertex, say point  $A$ , is as follows. Through point  $\bar{A}_2$  parallel to trace  $\bar{k}_1$  draw the horizontal  $\bar{A}_2N_2$  coincided with  $\Pi_2$ . Through  $N_2$  draw its frontal projection  $N_2A_2$  parallel to ground line  $Ox$ . The projection  $A_2$  of point  $A$  is found by passing a perpendicular from  $\bar{A}_2$  to trace  $l_2$  and extending it to intersect the frontal projection of the horizontal. Once  $N_2$  has been found, the projection  $N_1$  may be determined. Then through  $N_1$  the horizontal projection of the horizontal is drawn and on it, with the help of a projector, the

horizontal projection  $A_1$  of point  $A$  is determined. The other five vertices of the hexagon  $ABCDEF$  are found in a similar way.

The horizontal and frontal projections of a hexagon situated in plane  $\alpha$  are found by joining the corresponding points by straight lines.

7. Construct in plane  $\alpha$  a circle of a radius  $r=17$  mm with its centre at point  $C$  (Fig. 275).

A horizontal  $CN$  is drawn in plane  $\alpha$  through point  $C$ . The frontal trace  $N$  of this horizontal is then made to coincide with plane  $\Pi_1$ . After that through points  $X_\alpha$  and  $\bar{N}_2$  the frontal trace  $\bar{l}_2$  of plane  $\alpha$  coinciding with  $\Pi_2$  is drawn. Through  $\bar{N}_2$  parallel to  $k_1$  a horizontal in coincidence with  $\Pi_1$  is drawn. Next, from point  $C_1$  a perpendicular is dropped to trace  $k_1$  of plane  $\alpha$  to intersect this coincided horizontal in point  $\bar{C}_1$ . From point  $\bar{C}_1$  as the centre, a circle of radius  $r$  is described. Two pairs of mutually perpendicular diameters,  $\bar{A}_1\bar{B}_1 \perp \bar{D}_1\bar{E}_1$  and  $\bar{K}_1\bar{L}_1 \perp \bar{R}_1\bar{F}_1$ , are drawn, diameter  $\bar{A}_1\bar{B}_1$  being parallel to trace  $k_1$  and diameter  $\bar{K}_1\bar{L}_1$  parallel to trace  $\bar{l}_2$ .

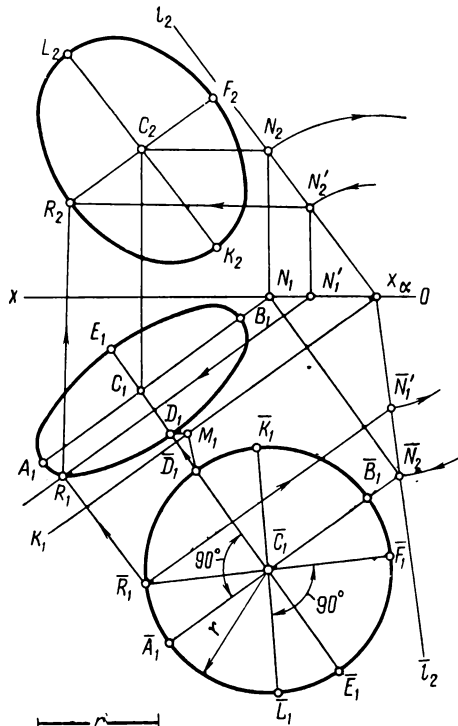


Fig. 275

Since diameter  $AB$  of the circle is parallel to plane  $\Pi_1$ , it will be projected onto  $\Pi_1$  in its true length. Therefore  $A_1B_1$  ( $A_1B_1 = \bar{A}_1\bar{B}_1$ ) is the major axis of the ellipse which is the projection of the circle on  $\Pi_1$ . The diameter  $KL$  parallel to plane  $\Pi_2$  determines the position and length of the major axis of the ellipse which is the frontal projection of the circle. The minor axes of these ellipses are perpendicular to the major axes and are determined by the projections of the diameters  $DE$  and  $RF$ .

After restoration of points of the circle from the coincided position to plane  $\alpha$  points  $A_1$  and  $B_1$  will lie on the horizontal projection of the horizontal  $CN$ , each at a distance equal to radius



$r$  from point  $C_1$ . Points  $K_2$  and  $L_2$  will lie on the frontal projection of the frontal drawn through point  $C_2$ , also at a distance equal to radius  $r$  from point  $C_2$ .

Projections  $A_1B_1$  and  $K_2L_2$  are equal in length to diameter of the circle. The projections of the diameters  $DE$  and  $RF$  on planes  $\Pi_1$  and  $\Pi_2$  will be perpendicular to the corresponding projections of diameters  $AB$  and  $KL$ , namely,  $D_1E_1 \perp A_1B_1$  and  $R_2F_2 \perp K_2L_2$ .

To find the projection  $D_1E_1$ , which is the minor axis of the ellipse in plane  $\Pi_1$ , it is sufficient to construct the horizontal projection of only one extreme point such as  $D$  of the diameter  $DE$ . The length of the minor axis of the ellipse in plane  $\Pi_2$  may be determined by restoring to plane  $\alpha$  one of the ends of the diameter  $RF$ , for example point  $R$ , from the coincided position.

Point  $D_1$  may be found with the help of frontal  $DM$ ,  $\overline{D_1M_1}$  being parallel to  $\overline{l_2}$ . Point  $E_1$  is obtained by marking distance  $C_1D_1$  on the minor axis  $D_1E_1$  but on the opposite side of major axis  $A_1B_1$ .

The point  $R_2$  in plane  $\Pi_2$  is constructed with the help of the horizontal  $RN$ . Point  $F_2$  lies on the opposite side of the major axis of ellipse  $K_2L_2$  at the same distance from point  $C_2$  as the point  $R_2$ .

Once the axes have been found the ellipses, which are the complete horizontal and frontal projections of a circle of radius  $r$  contained in plane  $\alpha$ , may be drawn.

## § 29. Revolution with Parallel Transfer of Projections

Sometimes the views obtained by revolving projections about axes perpendicular to planes of projection are found to lie one on top of the other. This can be avoided by applying another method (see Fig. 276). In this drawing by means of two successive revolutions about axes respectively perpendicular to planes of projection  $\Pi_1$  and  $\Pi_2$ , the plane of triangle  $ABC$  is made to lie parallel to the horizontal plane of projection  $\Pi_1$ .

The triangle is brought into the position  $ABC$  perpendicular to plane  $\Pi_2$  by the first revolution. The construction is carried out with the help of the horizontal  $CF$  which is rotated about an axis perpendicular to plane  $\Pi_1$  until it is perpendicular to the frontal plane of projection  $\Pi_2$ .

It is convenient to shift the horizontal projection  $\overline{A_1B_1C_1}$  to the right side of the drawing where there is space. This corresponds to rotating it about an axis perpendicular to the plane of projection  $\Pi_1$  and changes neither its form nor size.

The frontal projection  $\overline{A_2B_2C_2}$  is then obtained in the usual way, i. e., from points  $\overline{A_1}$ ,  $\overline{B_1}$ ,  $\overline{C_1}$  projectors are drawn perpendicular to ground line  $Ox$  to intersect the corresponding lines drawn through

points  $A_2, B_2, C_2$  parallel to ground line  $Ox$ . The points  $A_2, B_2$  and  $\bar{C}_2$  are then joined by the straight line  $\bar{A}_2\bar{B}_2\bar{C}_2$ .

By means of a second revolution about an axis perpendicular to plane  $\Pi_2$  the triangle  $\bar{A}\bar{B}\bar{C}$  is brought to the position  $\bar{\bar{A}}\bar{\bar{B}}\bar{\bar{C}}$  parallel to plane  $\Pi_1$ . The frontal projection  $\bar{\bar{A}}_2\bar{\bar{B}}_2\bar{\bar{C}}_2$  is then parallel to ground line  $Ox$ .

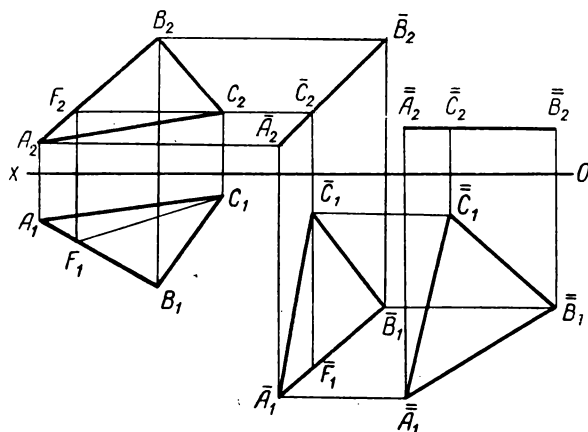


Fig. 276

This projection  $\bar{\bar{A}}_2\bar{\bar{C}}_2\bar{\bar{B}}_2$  is shifted by a parallel transfer to a convenient position at the right of the drawing. Projectors are drawn through points  $\bar{\bar{A}}_2, \bar{\bar{B}}_2$  and  $\bar{\bar{C}}_2$  and through points  $\bar{A}_1, \bar{B}_1, \bar{C}_1$  lines parallel to ground line  $Ox$  are passed. Their intersections give points  $\bar{\bar{A}}_1, \bar{\bar{B}}_1$  and  $\bar{\bar{C}}_1$ . Triangle  $\bar{\bar{A}}_1\bar{\bar{B}}_1\bar{\bar{C}}_1$  is a true view of triangle  $ABC$ .

### § 30. Combined Transformations of Projections

So far we have examined the revolution of various geometrical elements about axes perpendicular or parallel to one of the planes of projection, the rotation being made about axes either parallel to plane  $\Pi_1$  or plane  $\Pi_2$ . Moreover, this rotation was carried out not through any definite angle but only up to the one particular position when the figure being drawn becomes parallel to a plane of projection.

Sometimes it becomes necessary to revolve geometrical elements through a definite angle about oblique axes or axes parallel to one of the planes of projections.

In such cases it is common practice to resort to combined transformations. The axis of revolution in that case is first positioned perpendicular to one of the planes of projection. If we

revolve a geometrical element through a given angle about an axis in such a position we find the projection of the element in the system of principal planes of projection  $\Pi_1$ - $\Pi_2$ .

Let us consider some examples.

**Example 1.** It is required to revolve point  $A$  about axis  $MN$  parallel to plane  $\Pi_1$  through angle  $\alpha=90^\circ$  in a counter-clockwise direction (Fig. 277).

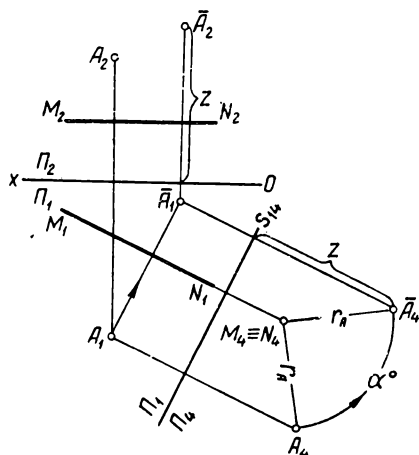


Fig. 277

First, the plane of projection  $\Pi_2$  for  $\Pi_4$  is changed so that the axis of revolution is projected onto plane  $\Pi_4$  as a point. To ensure this plane  $\Pi_4$  is drawn perpendicular to both line  $MN$  and plane  $\Pi_1$ . This is possible since  $MN$  is parallel to  $\Pi_1$ . The axis  $MN$  is projected onto plane  $\Pi_4$  as point  $M_4 \equiv N_4$ , and point  $A$  as point  $A_4$ .

We then revolve point  $A$  through the given angle  $\alpha$  in the system of planes of projection  $\Pi_1$ - $\Pi_4$ . The projection  $A_4$  will move into position  $\bar{A}_4$  and the projection  $A_1$  into position  $\bar{A}_1$ . This is indicated by arrows.

Then we find the projection  $\bar{A}_2$  by drawing through  $\bar{A}_1$  a projector perpendicular to ground line  $Ox$  and laying off on it upwards from  $Ox$  the value of  $z$  for point  $\bar{A}$ .

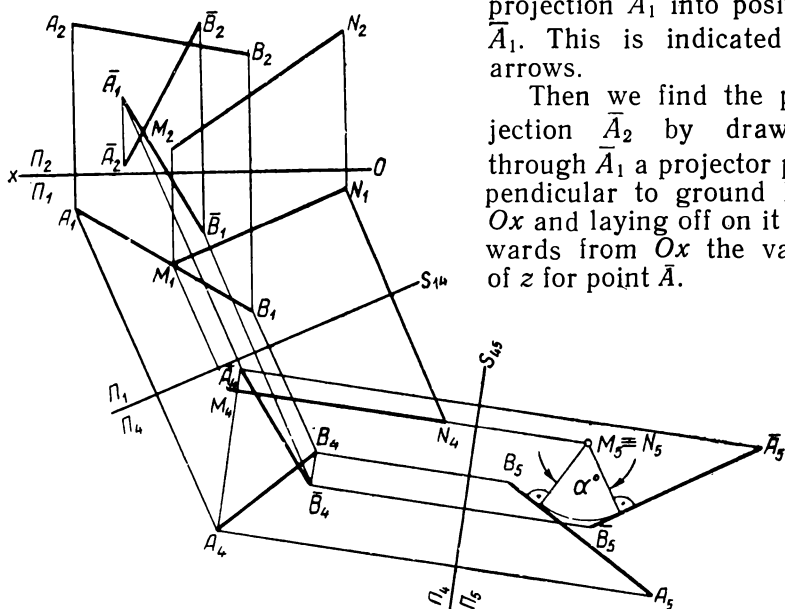


Fig. 278

The constructed projections  $\bar{A}_1$  and  $\bar{A}_2$  are the projections of the given point  $A$  after it has been transferred to the required position  $\bar{A}$ .

**Example 2.** Given two oblique lines  $AB$  and  $MN$ , it is required to revolve line  $AB$  about line  $MN$  through an angle  $\alpha = 60^\circ$ , clockwise (Fig. 278).

By successive substitution of the planes of projection  $\Pi_1$  and  $\Pi_2$  by the planes  $\Pi_4$  and  $\Pi_5$  we transform the projection of the axis of revolution  $MN$  into a point contained in the system of planes of projection  $\Pi_4$ - $\Pi_5$ . Line  $AB$  should also be constructed in this system of planes of projection.

Having done that we revolve line  $AB$  about axis  $MN \perp \Pi_5$  through the angle  $\alpha$ . Lines  $\bar{A}_5\bar{B}_5$  and  $\bar{A}_4\bar{B}_4$  are projections on  $\Pi_5$  and  $\Pi_4$  of line  $AB$  when it is turned into the required position  $\bar{AB}$ . The projections  $\bar{A}_1\bar{B}_1$  and  $\bar{A}_2\bar{B}_2$  of line  $\bar{AB}$  in the given system of planes of projection  $\Pi_1$ - $\Pi_2$  are found by means of reverse projections.

### § 31. The Use of Related Conformity in Transforming Projections

1. It is possible to establish a definite relationship between the position of a geometrical element when it has been brought into coincidence with a plane of projection and the projection of that element on the given plane of projection.

For example, in Fig. 275 the horizontal projection  $C_1$  corresponds to point  $\bar{C}_1$  which coincides with plane  $\Pi_1$ . Projection  $A_1B_1$  corresponds to the coincided diameter  $\bar{A}_1\bar{B}_1$ . The ellipse  $A_1B_1D_1E_1$  corresponds to the circle with centre  $\bar{C}_1$ . Line  $D_1M_1$  corresponds to line  $\bar{D}_1\bar{M}_1$  and so on.

This relationship is called *related conformity* and is studied in the course of projective geometry, in which it is proved that between the geometrical element itself and each of its orthographic projections and also between the orthographic projections of the element on planes  $\Pi_1$  and  $\Pi_2$ , a related conformity may always be established.

The following invariants exist in related conformity (see Fig. 275):

1) One and only one point of a plane corresponds to each point of another plane. For example, only point  $C_1$  corresponds to point  $\bar{C}_1$ .

2) To each line contained in a plane there is only one corresponding line in another plane. For example, only line  $D_1M_1$  corresponds to line  $\bar{D}_1\bar{M}_1$ .

3) Each pair of related lines intersects on a line called the *axis*

of alliance or the axis of conformity. Points contained in this axis are called *duplicate points* since they correspond to themselves.

4) The duplicate point of two related lines parallel to the axis of conformity lies at infinity (for example,  $\bar{A}_1\bar{B}_1$  and  $A_1B_1$ ).

5) For a point contained in a line there is one corresponding point contained in any other related line. For example, point  $C_1$  on line  $A_1B_1$  corresponds to point  $\bar{C}_1$  on line  $\bar{A}_1\bar{B}_1$ .

6) Two parallel lines in one plane correspond to two parallel lines in another plane.

7) If a part of a line in one plane is divided into some ratio by a point, the part of a line related to it in another plane is divided in the same ratio by the point corresponding to the first point.

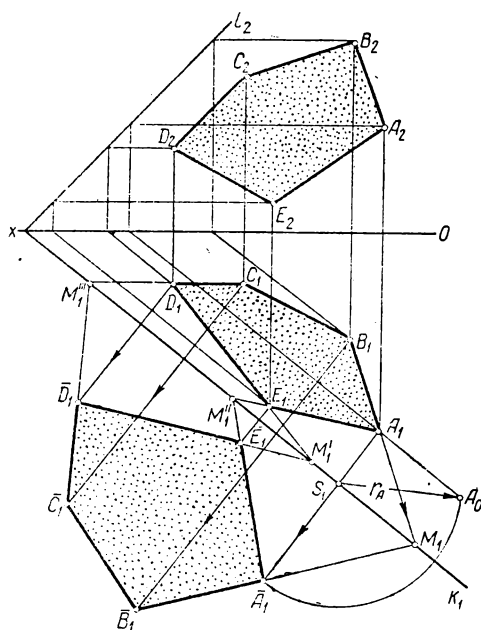


Fig. 279

2. The use of related conformity for the transformation of projections allows a considerable decrease in the number of auxiliary lines in the drawing as compared with the more usual constructions.

For example, it is required to find the true size of the pentagon  $ABCDE$  contained in plane  $\alpha$

(Fig. 279). When constructing this pentagon, after it has been made to coincide with plane  $\Pi_1$ , we shall use the properties of related conformity.

Vertex  $A$  of the pentagon is made to coincide with plane  $\Pi_1$  by revolution about trace  $k_1$  of plane  $\alpha$ . This is done in the usual way, i. e., with the help of right triangle  $S_1A_1A_0$ , the true size of the radius of revolution  $r_A$  of point  $A$  is determined. This length is laid off from point  $S_1$  on the perpendicular to  $k_1$ . We then obtain  $\bar{A}_1$ , that is the position of point  $A$  when it has been made to coincide with plane  $\Pi_1$ .

Having assumed trace  $k_1$  as the axis of conformity of the figure  $A_1B_1C_1D_1E_1$  to be transformed into the related figure  $\bar{A}_1\bar{B}_1\bar{C}_1\bar{D}_1\bar{E}_1$  in coincidence with plane  $\Pi_1$  and assuming that point  $\bar{A}_1$  corres-

ponds to point  $A_1$ , the subsequent construction is carried out by using related conformities.

Extend line  $A_1B_1$  to intersect the axis of conformity  $k_1$  in the duplicate point  $M_1$ . Join  $M_1$  with point  $\bar{A}_1$  by line  $\bar{A}_1M_1$  which is related to line  $A_1M_1$ . Draw a perpendicular from  $B_1$  to  $k_1$ , i. e., in the direction of conformity of  $B_1\bar{B}_1$ . The intersection of this perpendicular with line  $A_1M_1$  gives point  $\bar{B}_1$  related to point  $B_1$ .

Point  $\bar{E}_1$  related to point  $E_1$  is constructed in a similar manner with the help of line  $A_1M_1''$  and the line  $\bar{A}_1\bar{M}_1''$  related to it. Point  $\bar{E}_1$  is found at the intersection of a perpendicular drawn from  $E_1$  to  $k_1$  with line  $\bar{A}_1M_1''$ .

By similar constructions points  $\bar{D}_1$  and  $\bar{C}_1$  are also found. The pentagon  $\bar{A}_1\bar{B}_1\bar{C}_1\bar{D}_1\bar{E}_1$  constructed in this way is related to pentagon  $A_1B_1C_1D_1E_1$  and is the pentagon  $ABCDE$  when it is brought into coincidence with plane  $\Pi_1$ .

### Problems and Exercises

1. In what type of projecting plane does a point move when it is revolved about a) a horizontal, b) a frontal?

2. Describe the procedure for constructing a multiview drawing of a point made to coincide with the frontal plane of projection by revolving it about a line contained in plane  $\Pi_2$ .

3. What line should be taken as the axis to revolve a plane figure: a) into plane  $\Pi_1$ , b) into plane  $\Pi_2$ ?

4. Why is a horizontal of a plane located parallel to the horizontal trace of plane, and its frontal — parallel to its frontal trace after the plane has been rotated to lie in a plane of projection?

5. Given the position of a point rotated into a plane of projection, from an oblique plane, how can the projection of this point in the given oblique plane be determined?

6. Rotate vertex  $B$  of triangle  $ABC$  about the frontal  $AK$  until it lies at the same distance from plane  $\Pi_2$  at which the frontal lies (see Fig. 252). (See § 27.)

7. Rotate point  $A$  about the line  $MN$  located in  $\Pi_2$  into the frontal plane of projection (Fig. 280). (See § 27.)

8. Make the frontal-projecting plane  $\alpha$  coincide with plane  $\Pi_2$ .

9. Make the plane  $\alpha$  given by point  $A$  and ground line  $Ox$  coincide with plane  $\Pi_1$ .

10. In plane  $\alpha$  draw a line  $CD$  at 20 mm from  $AB$  and parallel to  $AB$  which is contained in plane  $\alpha$  (Fig. 281).

Hint. This problem may be solved by the method of coincidence (see § 28).

11. The trace  $l_2$  of plane  $\alpha$  is given. The angle between traces  $l_2$  and  $k_1$  in space is  $75^\circ$ . Construct in this plane an isosceles

triangle  $ABC$  with a base  $AB=40$  mm and altitude  $H=50$  mm (Fig. 282). (See Fig. 274 and related text.)

12. Construct a regular hexagonal pyramid with its base in a given plane  $\alpha$ . The side of the base is 30 mm, the altitude — 50 mm. The centre of the base lies at point  $C$  (Fig. 283).

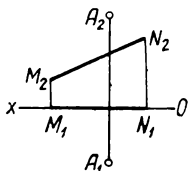


Fig. 280

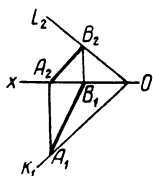


Fig. 281

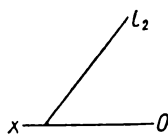


Fig. 282

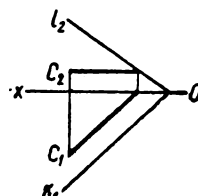


Fig. 283

Hint. When determining the projection of the vertex of the pyramid it is necessary to find the true length of any limited part of the line of altitude of the pyramid.

13. Find the true size of triangle  $ABC$  which lies in a plane parallel to ground line  $Ox$  and inclined to planes of projection  $\Pi_1$  and  $\Pi_2$ .

14. On an oblique plane construct a cylinder the diameter of which is 40 mm and the altitude — 70 mm.

Hint. The solution is similar to that given in the hint to question 12.

## Chapter VII

### REPRESENTATIONS OF GEOMETRICAL BODIES AND THE DEVELOPMENTS OF SURFACES

#### § 32. Multiview Drawings of Geometrical Bodies and Projections of Points and Lines on their Surfaces

1. Three projections of various geometrical bodies are shown in Fig. 284. These bodies are a tetragonal parallelepiped (Fig. 284, *a*), triangular and hexagonal prisms (Fig. 284, *b*, *c*), triangular and tetragonal pyramids (Fig. 284, *d*, *e*), the frustum of a tetragonal pyramid (Fig. 284, *f*), a cylinder (Fig. 284, *g*), a cone (Fig. 284, *h*), a sphere (Fig. 284, *i*), and an annular torus or anchor ring (Fig. 284, *j*).

All these bodies are depicted in the most convenient position determining their projections. That is to say, their axes, edges and faces are projected onto certain of the planes of projection in their true size and onto others as points and lines. Thus, for instance, all the vertical edges and the front face of the right parallelepiped (Fig. 284, *a*) are projected onto plane  $\Pi_2$  in their true size and on plane  $\Pi_1$  as points  $A_1 \equiv A'_1$ ,  $B_1 \equiv B'_1$ ,  $C_1 \equiv C'_1$ ,  $D_1 \equiv D'_1$ , while the face is projected as line  $A_1 D_1 \equiv A'_1 D'_1$ .

When working out and reading the multiview drawings of geometrical bodies special attention should be paid to the conformity of the dimensions repeated in pairs on each of the three pairs of planes of projection, i. e., on  $\Pi_1$  and  $\Pi_2$  (parallel to ground line  $Ox$ ), on  $\Pi_2$  and  $\Pi_3$  (parallel to line  $Oz$ ) and on  $\Pi_1$  and  $\Pi_3$  (parallel to line  $Oy$ ). For example, in Fig. 284, *c* the dimension  $h$ , taken in plane  $\Pi_1$  parallel to ground line  $Oy_1$ , is repeated in plane  $\Pi_3$  as the same dimension  $h$  parallel to ground line  $Oy_3$ . The altitude of the frontal and profile projections of the bodies, i. e., the dimensions parallel to line  $Oz$  in planes  $\Pi_2$  and  $\Pi_3$  are the same and so on.

The construction of multiview drawings of prisms, pyramids, cylinders and cones is usually begun on those planes of projection onto which their bases project in their true size, that is, as true view.

It should be taken into account that in a system of three planes



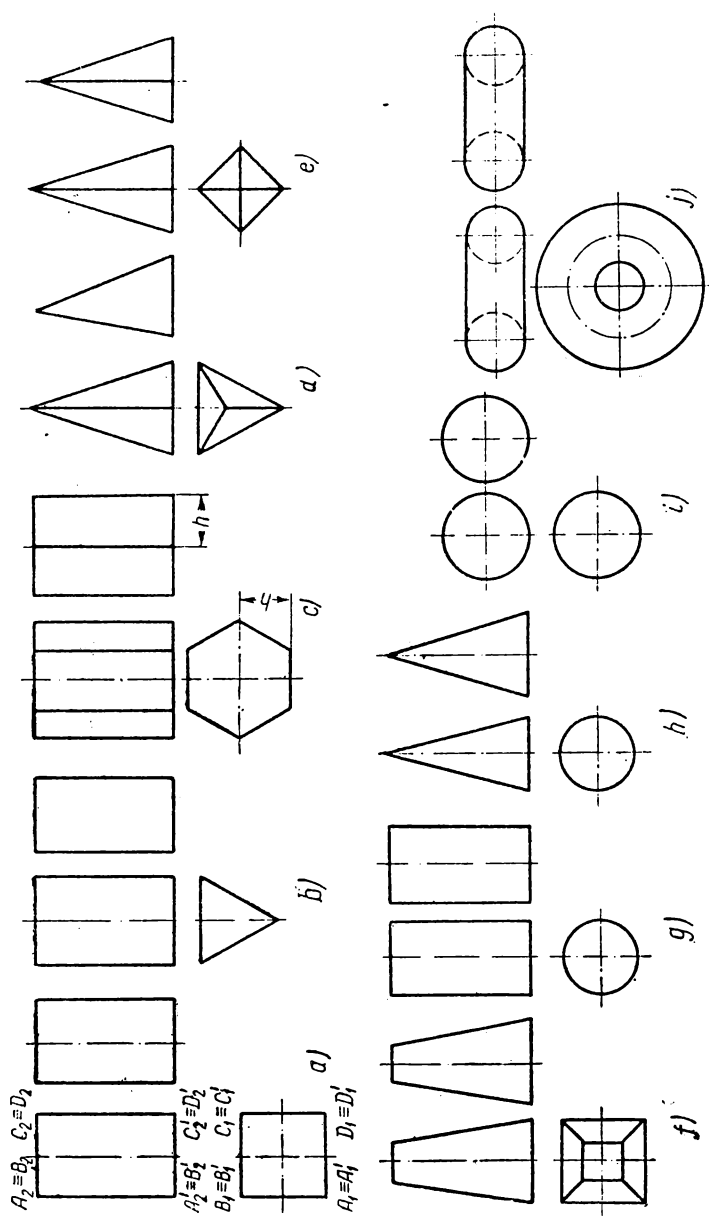


Fig. 284

of projection  $\Pi_1-\Pi_2-\Pi_3$  a cylinder is projected onto two of those planes as equal rectangles, and the cone — as isosceles triangles. The sphere on all three planes is projected as a circle whose diameter is equal to the diameter of the sphere.

2. The construction of projections of points contained in the surface of a geometrical body, as a rule, is carried out with the help of lines drawn through these points on the surface of the body.

It is possible to choose auxiliary lines in any direction. However, for simplicity these lines should be chosen so as to lie parallel to one of the edges of the given body.

For instance, it is required to construct at an altitude of 15 mm a point  $K$  on the right-hand front lateral, face of the tetrahedral regular pyramid frustum (Fig. 285).

This construction is carried out with the help of horizontal  $MK$  drawn on the face in question. For this purpose at an altitude of 15 mm from the ground line  $Ox$  we draw the frontal projection  $M_2K_2$  of the horizontal, the point  $M_2$  being contained in the frontal projection of the right-hand lateral edge  $BF$ . By drawing a projector from point  $M_2$  to intersect line  $B_1F_1$  we obtain point  $M_1$  through which we draw the projection  $M_1K_1$  of the horizontal parallel to the projection of edge  $A_1B_1$ , since  $A_1B_1$  is the horizontal trace of the plane of the right-hand front face of the pyramid. By drawing a projector through point  $K_2$  to intersect  $M_1K_1$  we find the horizontal projection  $K_1$  of point  $K$  contained in the surface of a pyramid and lying at a distance of 15 mm from plane  $\Pi_1$ . It is evident that any point contained in the horizontal  $MK$  will satisfy the conditions laid down.

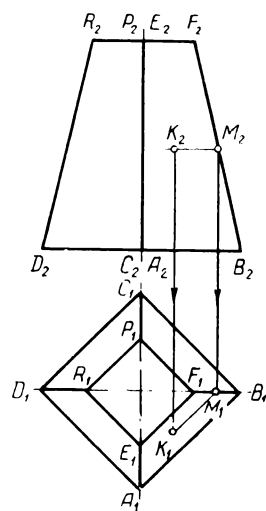


Fig. 285

3. The construction of points contained in the surface of a cone is carried out with the help of a circle or the generatrix, or generating line of the cone. Point  $A$  on the surface of the cone, Fig. 286, is constructed with the help of a circle of radius  $r$  ( $r=O_2D_2=O_1D_1$ ) and the point  $B$ , with the help of the generatrix  $SC$ . The sequence of the constructions in Fig. 286 is indicated by arrows. In this case the frontal projections  $A_2$  and  $B_2$  of points  $A$  and  $B$ , contained in the surface of the cone, were given.

Hint. The radius of the auxiliary circle, with the help of which point  $A_1$  is constructed, is not equal in length to line  $O_2A_2$ , which

is frequently and incorrectly assumed by students to be the true length of the required radius. The radius is equal to the length of  $O_2D_2$ . Line  $O_2A_2$  is the frontal projection of the radius of the given circle, inclined to plane  $\Pi_2$ . It is clear, when its projection  $S_1A_1$  on plane  $\Pi_1$  is drawn, that  $OA$  is projected onto plane  $\Pi_2$  as a foreshortened line  $O_2A_2$ .

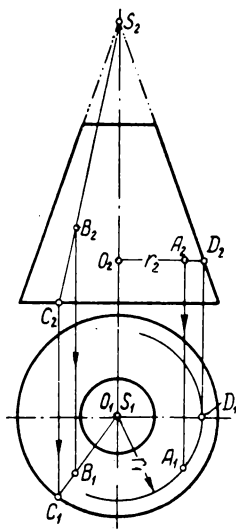


Fig. 286

4. In Fig. 287 a frontal projection  $A_2$  of point  $A$  has been constructed. This point lies on the surface of a sphere and the hori-

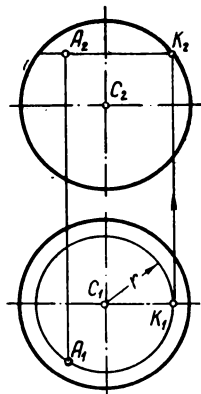


Fig. 287

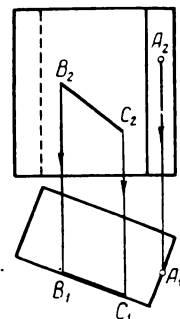


Fig. 288

zontal projection  $A_1$  of the point is given. An auxiliary circle has been drawn on the surface of the sphere through point  $A$ . This circle has a radius  $r$  which in the projection on  $\Pi_1$  is equal to the distance between points  $C_1$  and  $A_1$ , i. e., this is the true length of the radius. The frontal projection of the auxiliary circle is parallel to ground line  $Ox$  and passes through point  $K_2$  determined after point  $K_1$  has been determined, as shown by an arrow. By drawing a projector from point  $A_1$  to intersect the frontal projection of the auxiliary circle the frontal projection  $A_2$  of point  $A$  was found.

5. If a face of a body is perpendicular to the plane of projection, the construction of points on that face is carried out directly, without auxiliary lines. For instance, in order to construct the horizontal projection  $A_1$  of point  $A$  contained in the lateral face of a right tetragonal prism and given by the frontal projection  $A_2$  (Fig. 288), it is sufficient to draw a projector through point  $A_2$  to intersect the projection of the face on  $\Pi_1$  in point  $A_1$ .

The horizontal projection of line  $BC$  contained in another lateral

face of the same prism and its given frontal projection  $B_2C_2$  are found in a similar manner.

6. A triangular inclined pyramid frustum standing on plane  $\Pi_1$  is shown in Fig. 289. A part of this pyramid is cut out. The projection of this portion on plane  $\Pi_2$  is bounded by the lines  $1_23_25_2$  and  $2_24_26_2$ . By using the frontal projection let us construct the horizontal projection of the outline of the notch. Having found the horizontal projection  $2_1$  with the help of projector  $2_12_2$ , through point  $2_1$  we draw lines  $2_14_1$  and  $2_16_1$  respectively parallel to projections  $B_1D_1$  and  $B_1F_1$  of the edges of the base of the pyramid. At the intersection of line  $2_14_1$  with the projection of the lateral edge  $C_1D_1$  point  $4_1$  will be obtained. By drawing a line parallel to the projection  $D_1F_1$  of the third edge of the base of the pyramid through this point  $4_1$ , we find the projection  $4_15_1$  of another line making up the outline of the notch. We find point  $5_1$  on the projection  $4_15_1$  and point  $6_1$  on the projection  $2_16_1$  at the intersection of these lines with a projector drawn through point  $5_2=6_2$ . The horizontal projections of the other vertices of the notch, i. e., points  $1_1$  and  $3_1$ , contained respectively in the projections  $A_1B_1$  and  $C_1D_1$  of the lateral edges, are also found with the help of projectors, as shown by arrows. Having joined the points  $1_1$  and  $3_1$  by straight lines and with points  $6_1$  and  $5_1$ , respectively, we finish the construction of the horizontal projection of the notch left by the part cut out of the given geometrical body.

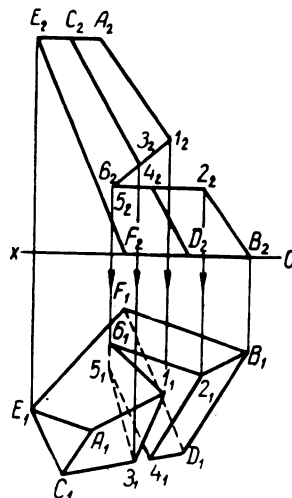


Fig. 289

### § 33. Development of the Surfaces of Geometrical Bodies

1. *The development of the surface of a body* is the figure obtained when the surface is made to coincide with a plane\*.

The development of the surface of a polyhedron consists in determining the true views of each of its faces in succession in the plane of the drawing. This is done in the same order as the faces are located in space. When developing oblique prisms and pyramids it is first necessary to determine the true size of each face with the help of one of the methods of transforming projections.

\* The term "the development of surfaces", particularly in automobile body designing offices, is frequently understood to mean the construction of the drawings of complicated double curve surfaces.

The development of the surface of right prisms, all the angles of the lateral faces being right angles, is carried out directly without any transformation of projections.

For example, in order to develop the lateral surface of a hexagonal right regular prism (Fig. 290) it is sufficient to draw a line and step off on it six parts  $AB$ ,  $BC$ , ...,  $FA$ , each of which is

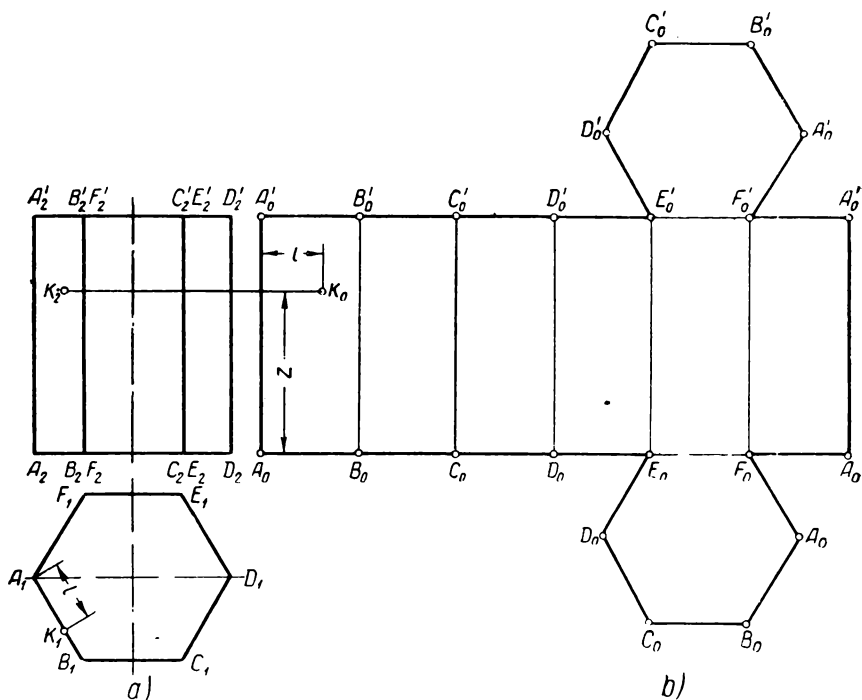


Fig. 290

equal to the side of the prism's base and then construct the rectangle  $A_0A_0A'_0A'_0$ . The height  $A'_0A_0$  of this rectangle is equal to the altitude of the prism.

In this developed surface the lateral edges of the prism  $A'_0A_0$ ,  $B'_0B_0$ , ... are situated perpendicular to the sides  $A_0A_0$  and  $A'_0A'_0$  of the rectangle, since all the angles of each face of the prism are right angles.

The complete development of the prism will be obtained if, adjoining the top and bottom sides of one developed face, we draw the top and bottom surfaces of the prism as projected onto plane  $\Pi_1$  in their true size in the form of regular hexagons.

The point  $K_0$  marked on the lateral face  $AA'B'B$  can easily be

found. On the development of this face point  $K_0$  lies at a distance from the line  $A_0B_0$  equal to the value of  $z$  for point  $K$  and to the right of line  $A'_0A_0$  at a distance  $A_1K_1=l$ .

2. The development of an oblique triangular prism (Fig. 291) becomes complicated because the true distance

between the edges of each lateral face cannot be read directly from the drawing. This distance may be determined by the transformation of projections. For example, by successive revolutions of the prism about two axes respectively perpendicular to the planes of

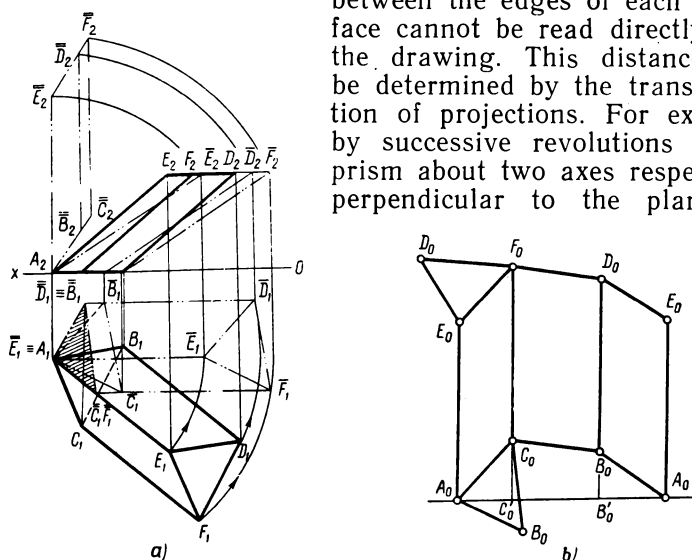


Fig. 291

projection  $\Pi_1$  and  $\Pi_2$ , the prism can be made to lie perpendicular to plane  $\Pi_1$ . Then on plane  $\Pi_1$  the prism will be projected as a triangle whose sides are equal in length to the true distances between the lateral edges.

The first revolution is about the axis perpendicular to  $\Pi_1$  and passing through the vertex  $A$ . The axis is not shown in the drawing. This brings the prism into the position  $\overline{A}\overline{B}\overline{C}\overline{D}\overline{E}\overline{F}$ . The projection in this position is drawn by single-dot-and-space lines. The lateral edges of a prism are now parallel to plane  $\Pi_2$  and so are projected onto it as true lengths.

By second revolution about the axis passing through the same vertex  $A$  but perpendicular to  $\Pi_2$ , we bring the prism into the vertical position  $\overline{\overline{A}}\overline{\overline{B}}\overline{\overline{C}}\overline{\overline{D}}\overline{\overline{E}}\overline{\overline{F}}$ . The projection of this position is represented by double-dot-and-space lines. The lateral edges of the prism will then be projected onto plane  $\Pi_2$  as segments  $A_2\overline{\overline{E}}_2$ ,  $\overline{\overline{B}}_2\overline{\overline{D}}_2$  and  $\overline{\overline{C}}_2\overline{\overline{F}}_2$  perpendicular to ground line  $Ox$ . On plane  $\Pi_1$  they will be projected as points  $A_1 \equiv \overline{\overline{E}}_1$ ,  $\overline{\overline{B}}_1 \equiv \overline{\overline{D}}_1$  and  $\overline{\overline{C}}_1 \equiv \overline{\overline{F}}_1$ . Joining these

points by straight lines we obtain the triangle which after the second revolution of the prism is projected onto  $\Pi_1$ . This triangle is indicated by hatching.

We now have all the dimensions required to develop the surface of the prism.

Let us draw a horizontal line (Fig. 291, *b*) and take on it a point  $A_0$  at random. From this point lay off the lengths of the sides of the hatched triangle in succession. We then obtain points  $C_0$ ,  $B_0$  and  $A_0$  through which we erect perpendiculars to the horizontal line. Then we lay off on these perpendiculars the true lengths of the lateral edges of the prism, taking into account the distance of the ground line  $Ox$  from points  $\bar{B}_2$  and  $\bar{C}_2$ .

Points  $A_0$ ,  $C_0$ ,  $B_0$ ,  $A_0$  and  $E_0$ ,  $F_0$ ,  $D_0$ ,  $E_0$  found in this way are joined by straight lines in the same order as on the prism. The figure  $A_0C_0B_0A_0E_0D_0F_0E_0$  is then the development of the sides of the prism. To obtain the complete development it is necessary to adjoin the bottom and top of the prism to the corresponding edges of one of the developed faces. The dimensions of the sides of these ends are taken directly from the drawing of the prism since they lie parallel to plane  $\Pi_1$  and are projected onto it in their true size. Thus,  $\triangle A_0B_0C_0 = \triangle E_0F_0D_0 = \triangle A_1B_1C_1$ .

3. Another method of developing the lateral surface of the same triangular prism is shown in Fig. 292. This method is known as the method of unwrapping. In this case the surface may be said to be "unwrapped". After converting the principal system of planes of projection  $\Pi_1$ - $\Pi_2$  by a new system  $\Pi_1$ - $\Pi_4$ , where plane  $\Pi_4$ , perpendicular to  $\Pi_1$ , is parallel to the lateral edges of the prism, we obtain the projection of a prism on plane  $\Pi_4$  in which these edges are seen in their true size as lines  $A_4E_4$ ,  $B_4D_4$  and  $C_4F_4$ . Then, in succession we revolve the lateral prism faces about the edges  $AE$ ,  $BD$  and  $CF$  until they are parallel to the plane of the drawing. That gives us the development of the lateral surfaces of the prism.

The vertices  $A$ ,  $B$ ,  $C$ , etc., during such rotations in space, will move in planes perpendicular to the lateral edges of the prism. In Fig. 292 the movement of points  $A_4$ ,  $B_4$  and  $C_4$  is shown by straight lines  $A_4A_0$ ,  $B_4B_0$  and  $C_4C_0$  drawn perpendicular to the lines  $A_4E_4$ ,  $B_4D_4$  and  $C_4F_4$ . To facilitate the construction on these perpendiculars of points  $A_0$ ,  $B_0$  and  $C_0$ , i. e., the vertices of the lower base of the prism rotated into the plane of the drawing, from  $A_0$  we describe an arc with radius  $A_0C_0 = A_1C_1$  to intersect line  $C_4C_0$  in point  $C_0$ . From  $C_0$  we describe an arc with radius  $C_0B_0 = C_1B_1$  to intersect line  $B_4B_0$  in point  $B_0$ . Finally, from point  $B_0$  an arc with radius  $B_0A_0 = B_1A_1$  is drawn to intersect  $A_4A_0$  in point  $A_0$ . Lines  $A_1B_1$ ,  $B_1C_1$  and  $C_1A_1$  give the true length of the sides of the prism's base.

Then, through points  $A_0$ ,  $B_0$  and  $C_0$ , we draw lines parallel to

the projections of the lateral edges of the prisms on plane  $\Pi_4$ . Next, we lay off on these lines, from points  $A_0, B_0$  and  $C_0$ , the true lengths of the edges. Joining up the points  $A_0, B_0, C_0, E_0, D_0$  and  $F_0$  in the corresponding order we obtain the development of the lateral surfaces of the prism.

4. The development of the lateral surface of a pyramid can be reduced to the construction of a series of triangles by finding

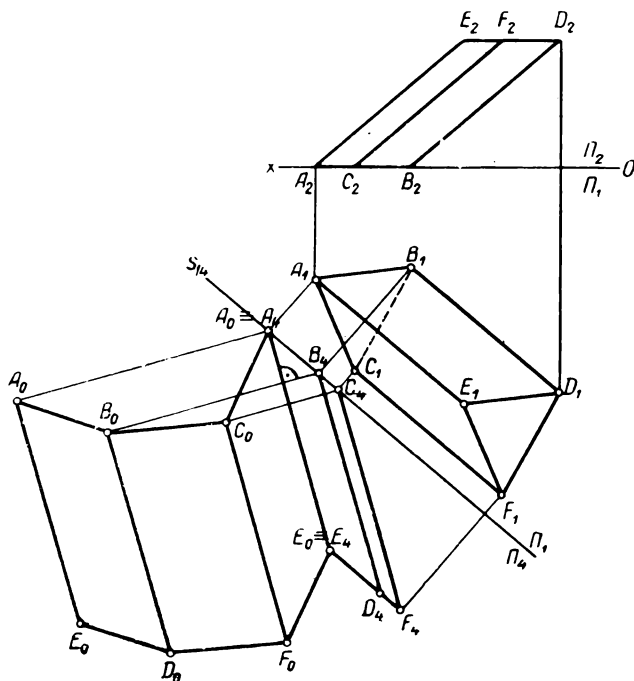


Fig. 292

their three sides. Each of these triangles will be a true view of the corresponding face of the pyramid.

The development of the surface of the oblique tetragonal pyramid  $SABCD$  (Fig. 293) is begun by determining the true lengths of the lateral edges. These edges are rotated about an axis passed through the vertex  $S$  of the pyramid and perpendicular to plane  $\Pi_1$ , to a position parallel to plane  $\Pi_2$ . All the lateral edges then are projected as true lengths. In Fig. 293 the construction is shown for the edge  $SA$ . Any convenient point is marked on the drawing. The triangle  $S_0A_0B_0$  is constructed by marking off distances equal to the true length of the sides of face  $SAB$ . Then the triangles



$S_0B_0C_0$  and so on are built up. After this we add to the development the sides of the base of the pyramid  $ABCD$ . These are taken from plane  $\Pi_1$  on which the base is projected as a true view. The construction of this base in the development is carried out with the help of two triangles,  $ABC$  and  $ADC$ , into which the base is first divided.

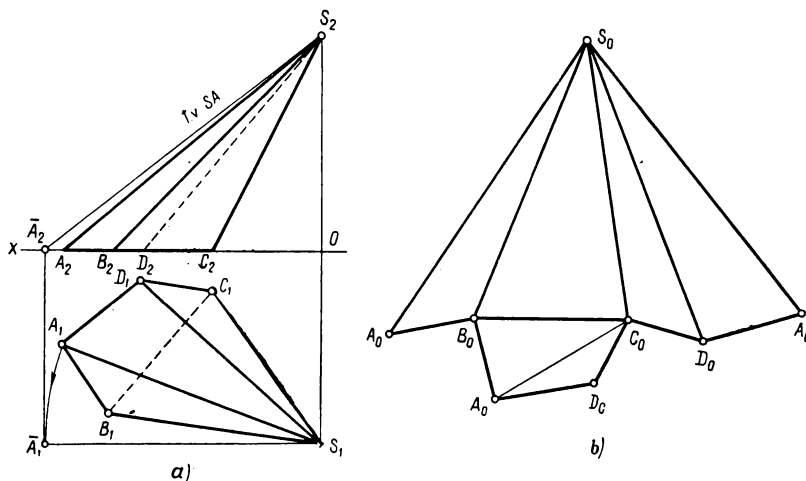


Fig. 293

5. The development of the lateral surface of a right circular cylinder is a rectangle the base of which is equal to the circumference  $2\pi r$  and the height of which is equal to the altitude of the cylinder.

The development of a cylindrical elbow pipe is shown in Fig. 294. The length of the development is equal to the length of the circumference of a normal cross-section, since the diameter of all portions of the pipe is the same. In order to determine the top and bottom outlines of the development the circumference of the upper portion is divided into 12 equal parts and through each dividing point we draw a generatrix on the cylindrical surface of the pipe. With the help of this on the surface of that part of the elbow to be developed, we draw the same number of generatrices. Then the central line of the development is also divided into 12 equal parts and through the points of division we draw perpendiculars on which we lay off upwards and downwards from the line distances equal to one half of the length of the corresponding generatrix of the pipe section. The curved outlines of the development or pattern are drawn through the points obtained.

6. The development of the lateral surface of an elliptical cylinder (Fig. 295) is carried out by "unwrapping" the surface.

First, the cylinder was projected onto plane  $\Pi_4$  which is perpendicular to plane  $\Pi_1$  and parallel to a generatrix. In this case all the generatrices of the cylinder will be projected onto plane  $\Pi_4$  in their true length. Then through the projections  $1_4, 2_4 \equiv 8_4, 3_4 \equiv 7_4, 4_4 \equiv 6_4$  and  $5_4$  of the eight points which divide the circumference into equal parts, perpendiculars are drawn to the projections of the generatrices on plane  $\Pi_4$ . It should be noted that the circumference is the projection of the lower end of the cylinder on plane  $\Pi_1$ .

The projections of the eight selected points, when the surface of the cylinder is being unwrapped, move onto the perpendiculars. The arcs are struck off successively on these perpendiculars beginning from point  $1_4$ . The arcs are equal in length to one division of the projection of the base on  $\Pi_1$  and determine the position of eight points on the development of the lower line of cylinder. Having drawn a continuous curve through these points and having constructed a second curve symmetrical to the first at a distance equal to the length of the generatrix of the cylinder, the development of the lateral surface of the given cylinder is completed.

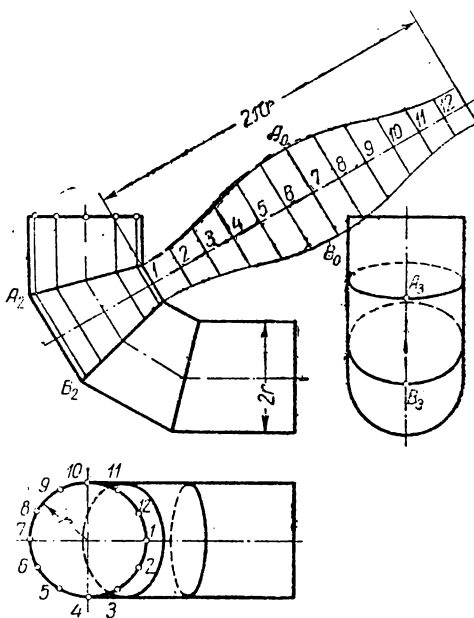


Fig. 294

7. The development of the surface of a right circular cone is a circular sector, the radius of which is equal to the length of the generatrix of the cone, and the central angle of which is determined by the formula:

$$\alpha^\circ = \frac{r \cdot 360^\circ}{l},$$

where  $r$  is the radius of the cone's base and  $l$  — the length of the generatrix.

A right circular cone truncated by the frontal-projecting plane

$\alpha$  and the development of its surface is shown in Fig. 296. After construction of sector  $S_0O_0I_2O_0$  whose arc is equal to the length of the circumference of the circular base of the cone and whose radius is equal to the length of the generatrix of the cone, the curve  $A_0B_0C_0D_0E_0F_0K_0L_0M_0N_0R_0V_0A_0$ , which is the development of the elliptical section is plotted on the developed surface.

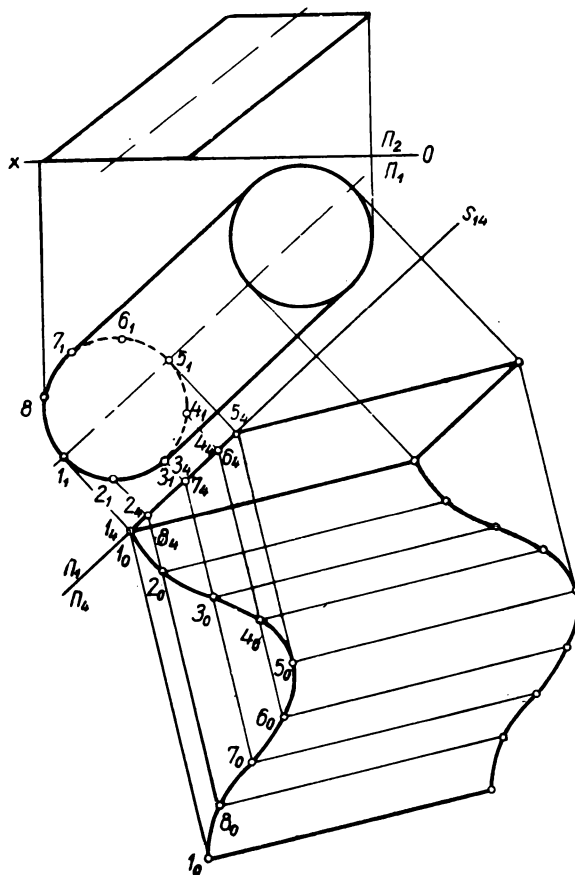


Fig. 295

The circle of the base of the cone is divided into 12 equal parts. Through these divisions we draw the 12 generatrices of the cone. The points  $A$  and  $K$  are contained in the limiting generatrices, or elements, of the cone. Both these elements are projected onto plane  $\Pi_2$  in their true length, so we shall find these points on the developed surface if we lay off from point  $S_0$ , on straight lines  $S_0O_0$  and  $S_0I_2O_0$ , a distance equal to  $S_2A_2$  and on line  $S_0\delta_0$  a distance

equal to  $S_2K_2$ . To determine the distances of the other points of the line of the ellipse to the vertex  $S$ , through their frontal projections we draw straight lines parallel to ground line  $Ox$  until they intersect one of the limiting generatrices of the cone, which corresponds to revolving them to a position parallel to plane  $\Pi_2$ . In Fig. 296 this is shown for point  $C_2$  only. Laying off these distances from  $S_0$  of the development, on the corresponding lines  $S_0I_0$ ,  $S_02_0$ ,  $S_03_0$ , etc., we obtain on each of the generatrices a point.

By drawing a curve through these points on the development we obtain the development of the cone cut by plane  $\alpha$ .

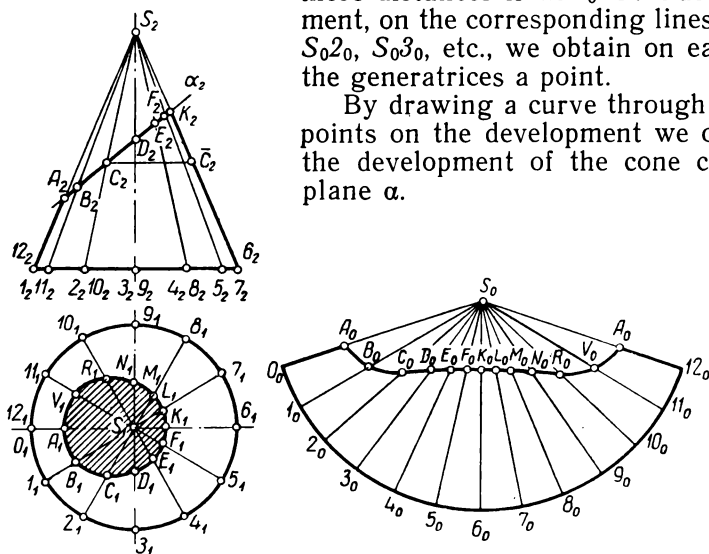


Fig. 296

8. When the generatrices of a frustum of a cone make angles close to  $90^\circ$  with its base and the apex lies beyond the limits of the drawing, an approximation of the development of the cone may be obtained, as shown in Fig. 297.

We construct an equilateral trapezoid  $KLMN$  whose sides  $KN$  and  $LM$  are equal to  $l$ , the length of the generatrix of the truncated cone. The top and bottom are respectively equal to the length of the circumferences of its base  $2\pi R$  and its top  $2\pi r$  ( $r$  is the radius of the top,  $R$  — the radius of the base). Divide sides  $KL$  and  $NM$  into 8 equal parts. Join the points obtained by straight lines 1-2, 3-4, etc.

Taking the sides  $KN$  and  $LM$  of the trapezoid as parts of the radius of a sector of a circle on the development, draw the arc of this sector passing through points  $M$  and  $N$ . The position of point  $A$  on the extension of line 7-8 is determined as follows: 1) from point 8 draw a perpendicular  $8B$  to side  $KN$  of the trapezoid; 2) construct the bisector  $8D$  of angle  $N8B$ ; 3) a distance equal to  $ND$  is laid off on line 7-8 downwards from point 8. The point  $A$

thus obtained is one of the points of the lower arc of the development of the cone.

The construction of any other point of this arc, for example, point  $D$ , lying on straight line 5-6, is carried out in the following sequence: 1) draw a perpendicular  $NR$  to line 5-6 from point  $N$ ; 2) from point  $R$  we draw a perpendicular  $RE$  to the side  $KN$ ;

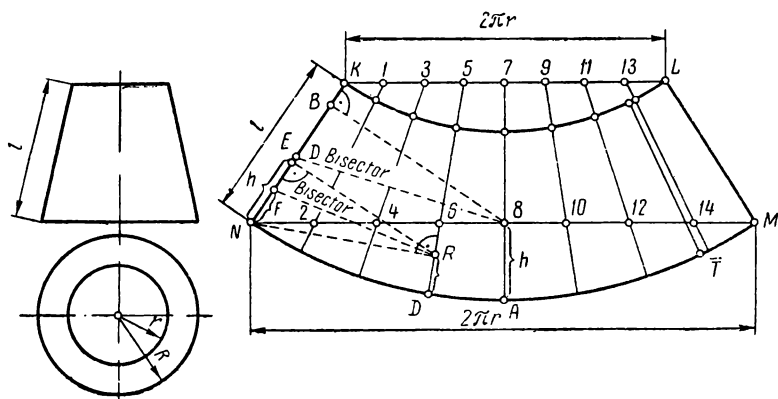


Fig. 297

3) draw the bisector  $RF$  of angle  $NRE$ . We find point  $D$  by stepping off a distance equal to  $NF$  on line 5-6 downwards from point  $R$ .

The points such as  $C$ , contained in arc  $KL$ , which is the development of the upper line of the cone, are found by laying off on the lines 7-8, 5-6, etc. from the points of the lower arc distances equal to the length  $l$  of the generatrix.

The figure  $KNDAMLCK$  constructed in this way is somewhat bigger than the true development. In order to obtain a more exact development, the divisions of the length of the circumference of the base and top of the cone should be laid off not on the chords but on the arcs  $KL$  and  $NM$ , each corresponding pair of points found should be joined by a straight line. In the drawing points  $A$  and  $T$  are shown. The figure  $KNDATSCCK$  will then be a more exact development of the cone.

9. The construction of the development of an oblique cone is similar to that of an oblique pyramid. A cone is shown in Fig. 298,  $a$ , and in Fig. 298,  $b$ , the development of its surface. This development is begun by dividing the base of the cone into any suitable number of parts, sufficiently small in length to ensure accuracy of the construction. Through the points chosen draw the generatrices  $SA$ ,  $SB$ ,  $SC$ , etc. The rest of the construction is that of the pyramid, shown in Fig. 293. Through the points

obtained a continuous curve  $F_0E_0D_0C_0B_0A_0N_0M_0L_0K_0F_0$  is drawn. This is the development of base line of the oblique cone.

So as to plot a point on this approximate development, say, point  $R$  lying on the surface of the cone, through point  $R$  draw a generatrix, then find this generatrix on the development and on it, from point  $S_0$  lay off the true distance from point  $R$  to the vertex  $S$

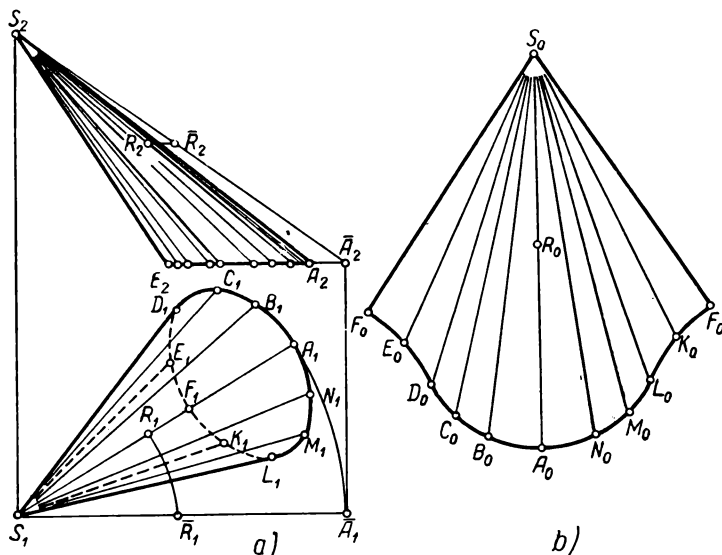


Fig. 298

of the cone. In Fig. 298 this distance is determined by the method of revolution.

10. The development of a sphere can only be approximate. Such a development is made by using either meridians or lines of latitude. Sometimes it is advantageous to develop one part of the sphere using the meridians and the other part by using lines of latitude.

If the sphere is divided up by meridians into a number of sections, or gores, then each section of the surface is "peeled" off the sphere as it were (Fig. 299, a). When the sphere is divided by lines of latitude parallel lines are drawn to divide the surface into a series of belts, or zones (Fig. 299, b). The middle equatorial belt is assumed to be a cylinder. The top and bottom belts are assumed to be cones and the medium belts, as frustums of cones. The development of the belts is then carried out according to the rules for developing a cylinder, a cone and frustums of cones.

The development of the sphere, shown in Fig. 300, is carried out

with the use of meridians. The horizontal projection of the sphere is divided into 12 equal parts. This gives the horizontal projections of the 12 portions ("gores") cut by meridional planes. To develop the first gore we draw a horizontal line, and from point  $A_0$  lay off on it  $A_0B_0$  equal to the chord of arc  $A_1B_1$ . Then through the middle  $A_0B_0$  we draw a perpendicular to it and step off on it points  $0_0, 1_0, 2_0, 3_0, 4_0$  and  $5_0$ . The distance between these points is equal

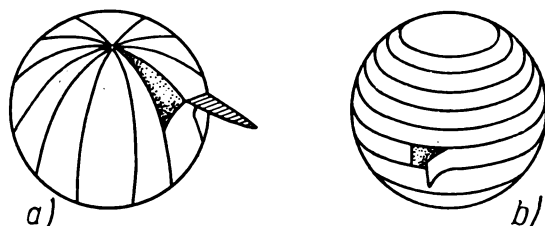


Fig. 299

respectively to the lengths of chords  $0_21_2, 1_22_2, 2_23_2, 3_24_2$  and  $4_25_2$ , into which a quarter of the circumference of the sphere was divided. These divisions are to be seen on the projection of the sphere surface on plane  $\Pi_2$ . Through points  $1_0, 2_0, 3_0$  and  $4_0$  we draw straight lines perpendicular to axis  $0_05_0$  and lay off on them to either side of the axis, at equal distances from it, points  $K_0$  and  $L_0, M_0$  and  $N_0, E_0$  and  $F_0, S_0$  and  $T_0$ .  $K_0L_0$  is equal to the chord of the arc  $K_1L_1$  drawn in plane  $\Pi_1$  through point  $I_1$  belonging to the first gore.  $M_0N_0$  is equal to the chord of the arc  $M_1N_1$ .  $E_0F_0$  is equal to the chord  $E_1F_1$  and, finally,  $S_0T_0$  is equal to the chord of arc  $S_1T_1$ .

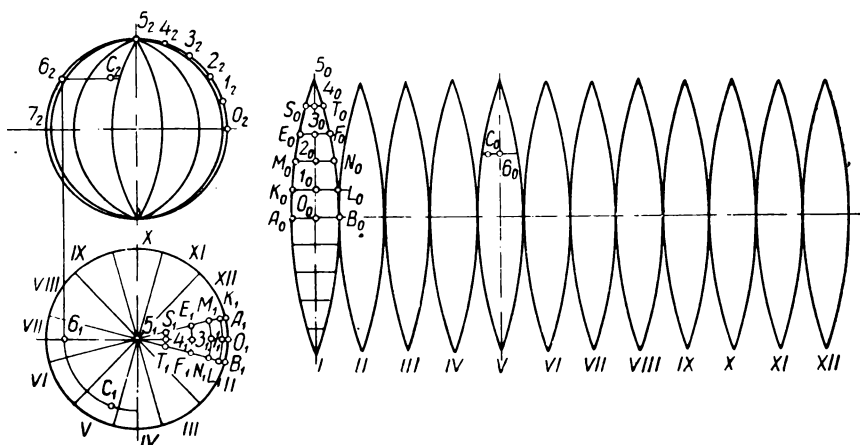


Fig. 300

The points  $A_0, K_0, M_0, E_0, S_0$  and  $5_0$  obtained in this way, and also points  $B_0, L_0, N_0, F_0, T_0$  and  $5_0$  are then joined by curves. The figure received is then an approximate development of the top half of the first gore.

The bottom half is symmetrical to the top half with respect to line  $A_0B_0$ . The remaining eleven parts of the sphere surface development are identical with the first gore.

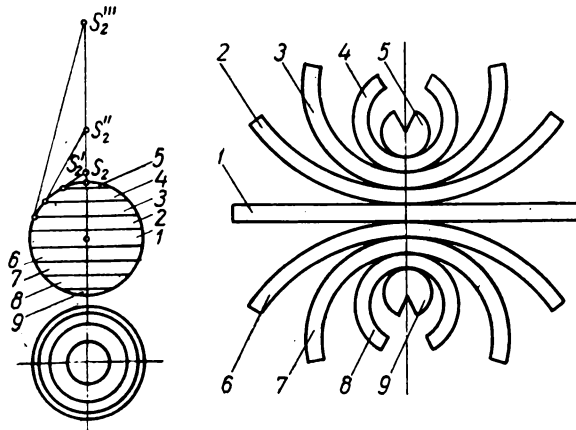
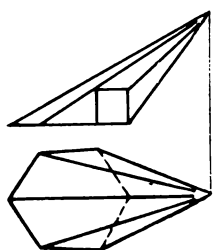


Fig. 301

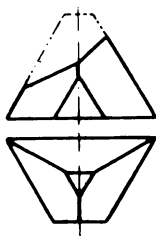
If it is required to find the position of the development of the sphere of point  $C$  situated in the top part of the fifth gore, it is first necessary to find the altitude of this point  $C$  in relation to the horizontal axis of development. For this purpose through point  $C$  a parallel is drawn on the surface of the sphere. Then, having measured the length of the chord of arc  $6_27_2$  we lay off this distance from line  $A_0B_0$  on the axis of development of the fifth gore. Then we draw a line parallel to the horizontal axis of development through point  $6_0$  and lay off on it, to the left of the vertical axis of the fifth gore, the distance from point  $C_1$  to the nearest boundary line of the fifth gore in the horizontal projection. The point  $C_0$  is then the required point.

The development of the sphere with the use of latitudes is shown in Fig. 301. The sphere is divided by a series of parallel planes into seven belts, or zones, and two spherical segments (5 and 9). The middle zone (1) is developed as a right circular cylinder. The development of zones 2, 3, 4, 6, 7 and 8 is carried out according to the method for developing right circular frustums of cones. The development of segments 5 and 9 is carried out in conformity with method of developing cones.

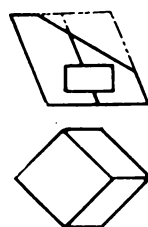




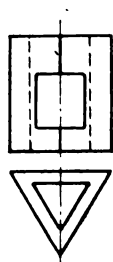
*Fig. 302*



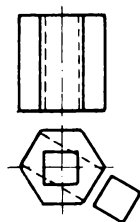
*Fig. 303*



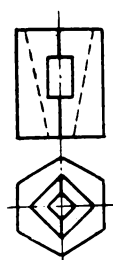
*Fig. 304*



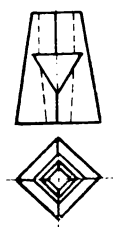
*Fig. 305*



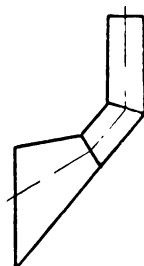
*Fig. 306*



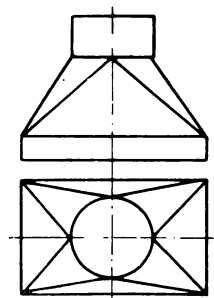
*Fig. 307*



*Fig. 308*



*Fig. 309*



*Fig. 310*

### Problems and Exercises

1. What is a development of the surface of a body?
2. What form is assumed by the development of the lateral surface a) of a right prism, b) of a right circular cylinder, c) of a right circular cone?

3. Construct the missing projections of the cut-outs and through holes of the bodies, depicted in Figs. 302-308. The construction should be carried out in three projections and on an augmented scale.

Hint. The procedure for constructing individual points is given in § 32.

4. Develop the surface of the pipe elbow (Fig. 309).

5. Develop the surface of the transition piece, shown in Fig. 310.

Hint. To carry out the development of the middle section, this surface should be divided into parts of triangular form.

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## *Chapter VIII*

### **INTERSECTION OF PLANES AND SURFACES**

#### **§ 34. Plane Sections of Polyhedrons**

1. The construction of plane sections of polyhedrons may be divided into two groups. The first group covers those cases when the edges of the polyhedron and the cutting or section planes are oblique. The second includes cases when the edges of the polyhedron or the section plane itself, or both, are perpendicular to planes of projection.

In any case, as the result of the intersection of the edges of the polyhedron by the cutting plane, a plane polygon is formed and the number of its vertices is equal to the number of the edges of the polyhedron.

When determining plane sections of the first group, the vertices of the polygonal section are found by using auxiliary planes, as was done when determining piercing points of a line in a plane and when finding the line of intersection of two planes.

The vertices of the polygons, received when solving problems of the second group, are constructed with the help of auxiliary lines drawn on the edges of the solids or simply by means of projectors.

2. The procedure for intersecting a triangular oblique pyramid by an oblique plane  $\alpha$ . This case belongs to the first group. The vertices of the triangle  $KLM$  which represents a section figure have been found as meeting points  $K$ ,  $L$  and  $M$  of the edges  $SA$ ,  $SB$  and  $SC$  with plane  $\alpha$ . For this purpose through each edge an auxiliary frontal-projecting plane is drawn. The line of intersection of each of these planes with the given plane  $\alpha$  is constructed. The intersection of each edge with the corresponding line gives the points in which the edges pierce, as is shown in Fig. 311.

For example, point  $K$  is found at the intersection of edge  $SA$  with line 1-2, in which plane  $\alpha$  intersects the auxiliary frontal-

projecting plane  $\beta$  passing through edge  $SA$ . The construction of point  $K$  is shown in Fig. 311, *b*.

The projection of the part  $SKLM$  of the pyramid which is cut away is shown by dashes and dots. The projections of triangle  $KLM$  of the section are shown by dots.

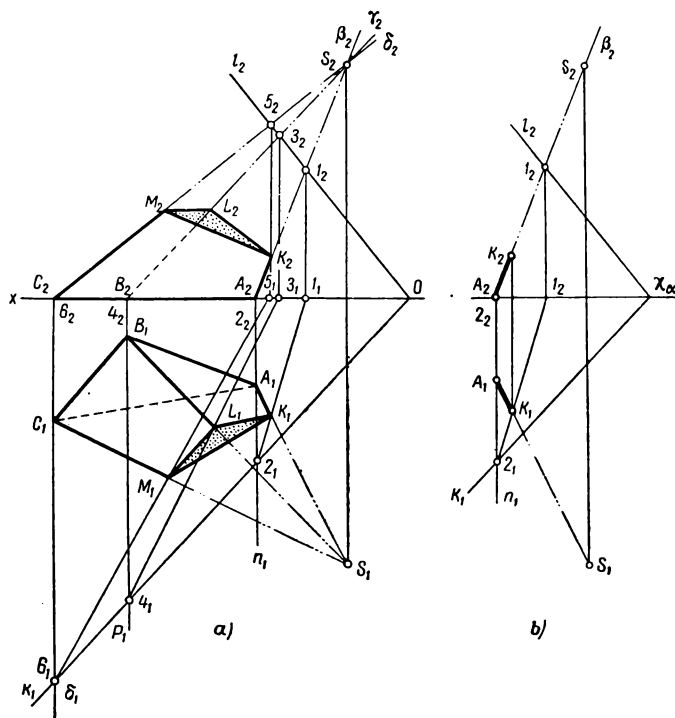
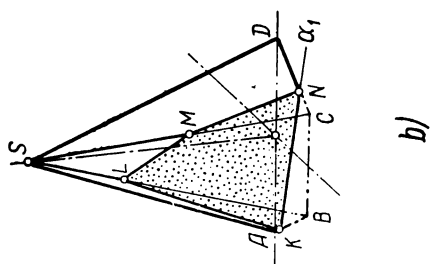
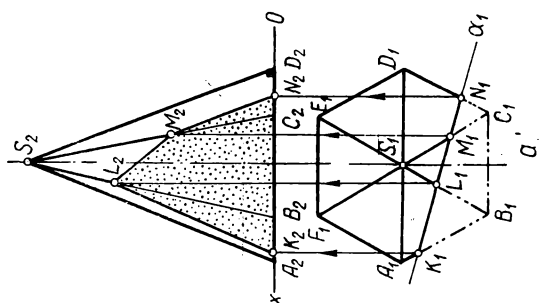
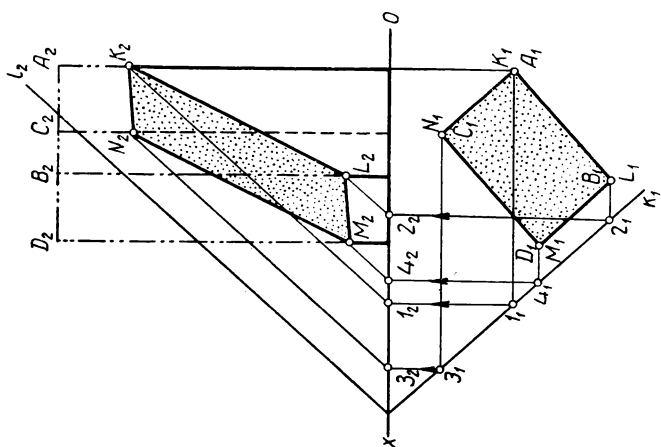


Fig. 311

3. The construction of plane figures, or sections, received when cutting polyhedra, in cases falling within the second group, is shown in Figs. 312 and 313.

In the first case (Fig. 312) the construction of the section  $KLMN$  of the right tetragonal prism by an oblique plane  $\alpha$  is reduced to the construction of the frontal projection  $K_2L_2M_2N_2$  of this figure using its horizontal projection  $K_1L_1M_1N_1$ , which coincides with the horizontal projection of the prism itself. This construction is done with the help of frontals of plane  $\alpha$  drawn through points  $K, L, M, N$ . At the intersection of the frontal projections of the frontals with the frontal projections of the respective edges of the prism points  $K_2, L_2, M_2$  and  $N_2$  are found. These points are then joined by straight lines. The tetragon  $K_2L_2M_2N_2$  is the frontal



projection of the section of the prism by plane  $\alpha$ . The portion of the prism which has been cut off is shown by dashes and dots and the projections of the tetragon section is shaded in by dots.

If the section plane is a projecting plane, then one projection of the section of the polyhedron coincides with the trace-projection of this plane which possesses collecting properties. The second projection is constructed with the help of the first using projectors. A right regular hexagonal pyramid cut by a horizontal-projecting plane  $\alpha$  is shown in Fig. 313, *a*. The points of intersection of the trace-projection  $\alpha_1$  of this plane with the projections of the lateral edges  $S_1B_1$  and  $S_1C_1$  and the edges of the base  $A_1B_1$  and  $C_1D_1$  determine the horizontal projections of the four vertices  $K_1$ ,  $L_1$ ,  $M_1$  and  $N_1$  of the section. By drawing projectors through these points to the corresponding projections  $S_2B_2$ ,  $S_2C_2$ ,  $A_2B_2$  and  $C_2D_2$ , points  $L_2$ ,  $M_2$ ,  $K_2$  and  $N_2$  are found. These are the frontal projections of the vertices of the section  $KLMN$ .

The way the pyramid is intersected by plane  $\alpha$  is illustrated by the axonometric representation, Fig. 313, *b*, in which the tetragon of the section is indicated by dots.

### § 35. Plane Sections of Curved Surfaces

1. The construction of plane sections of curved surfaces is similar to that of plane sections of polyhedra. In some cases the points contained in the intersection of a curved surface by a plane are constructed by drawing auxiliary lines, in others by using auxiliary planes.

Nevertheless, there are differences between the construction of plane sections of curved surfaces and the constructions of plane sections of polyhedra.

The difference is that the construction of points in a plane section of a curved surface is usually begun by finding on the figure so called *basic* or *special points* whose specific characteristics differ from those of other points. Special points include the highest and lowest points of a figure, points where the limiting elements are tangential to the curve of the section, the extreme points of the section, i. e., those nearest to and those farthest away from the observer, and certain other points.

2. Let us consider the intersection of a right circular cylinder by an oblique plane  $\alpha$  (Fig. 314, *a*).

Since plane  $\alpha$  is neither parallel to the axis of the cylinder nor perpendicular to it, the section will be an ellipse.

Let us pass a frontal plane  $\beta$  through the axis of the cylinder. This plane will cut the surface of the cylinder along its limiting generatrices or elements, and the plane  $\alpha$  along a frontal. The intersection of the frontal with the limiting elements will determine

two special points  $A$  and  $B$  contained in the ellipse of intersection.

Other special points are  $C$ , the point nearest to the observer, and  $D$ , the point farthest away. They are found with the help of the two frontal planes,  $\gamma$  and  $\delta$ , passing respectively through the front and rear generatrices of the cylinder. These planes also intersect plane  $\alpha$  in frontals. The intersection of the frontal projections of these frontals with the frontal projection of the axis of the

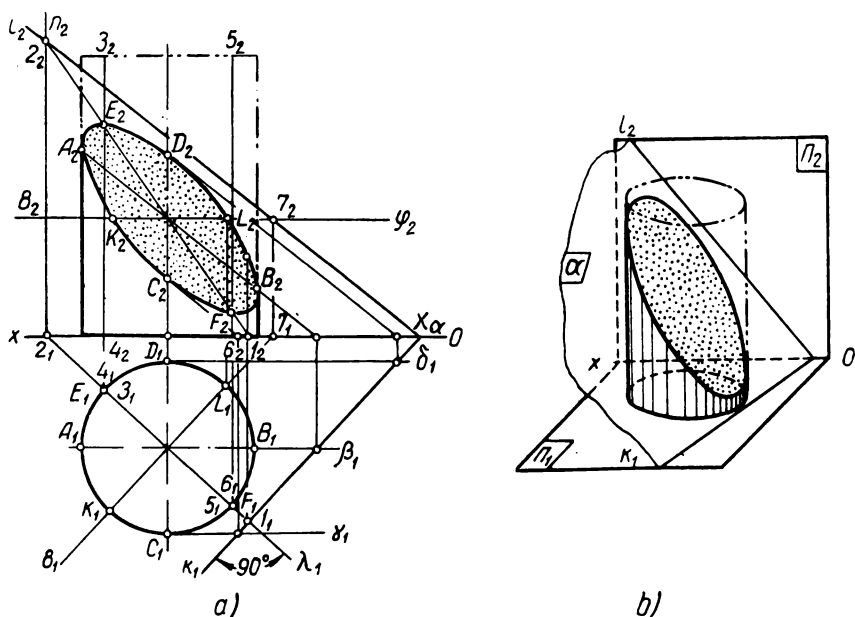


Fig. 314

cylinder determines the frontal projections  $C_2$  and  $D_2$  of these special points. The axis of the cylinder coincides with the projection on plane  $\Pi_2$  of the front and rear generatrices.

The highest point  $E$  and the lowest point  $F$  of the ellipse given by the intersection are found with the help of the horizontal-projecting plane  $\lambda$  passed through the axis of the cylinder perpendicular to trace  $k_1$ . The planes  $\lambda$  and  $\alpha$  intersect in line  $1-2$ . This is a line of maximum inclination of plane  $\alpha$  with respect to plane  $\Pi_1$ . Plane  $\lambda$  will intersect the cylinder along the generatrices  $3-4$  and  $5-6$ . The intersection of line  $1-2$  with lines  $3-4$  and  $5-6$  gives the special points  $E$  and  $F$ , respectively.

Besides the special points  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$  and  $F$ , the intermediate points  $K$  and  $L$  are also found. These points are constructed with the help of the horizontal plane  $\varphi$ . The section of this plane with the cylinder is a circle and it intersects plane  $\alpha$  in the horizontal

7-8. Points  $K_1$  and  $L_1$  lie at the intersection of the horizontal projection of the horizontal  $7_1-8_1$ , and the circle which coincides with the projection of the cylinder on  $\Pi_1$ . Having determined points  $K_1$  and  $L_1$ , with the help of projectors, we find points  $K_2$  and  $L_2$  on the trace  $\varphi_2$  of plane  $\varphi$ .

An axonometric representation of the cylinder cut by plane  $\alpha$  is shown in Fig. 314, *b*.

3. The intersection of a plane with a straight circular cone may give the following lines:

1) Two straight generatrices, when the cutting plane passes through the vertex of the cone.

2) A circle, when the cutting plane is perpendicular to the axis of cone.

3) An ellipse, when the plane intersects all the generatrices of the cone and yet is not perpendicular to the axis of the cone (Fig. 315).

4) A parabola, when the cutting plane is parallel to one of the generatrices (Fig. 316). In this case the angle  $\alpha$  between the cutting plane and the axis of the cone is the same as the angle  $\beta$  between the generatrix and the axis of the cone.

5) A hyperbola, when the cutting plane is parallel to any two generatrices of the cone or, as a special case, when this plane is parallel to the axis of the cone (Fig. 317). In the first case the angle  $\alpha$  is smaller than angle  $\beta$ . In the second, the angle  $\alpha$  is zero.

4. The construction of the section of a cone given by an oblique plane  $\alpha$  (Fig. 315). We begin by determining special points. The special points  $K$  and  $L$  are found with the help of the frontal plane  $\beta$  passed through the axis of the cone. This plane cuts the cone along the limiting elements  $SA$  and  $SB$ , and the plane  $\alpha$  in one of its frontals. At the intersection of

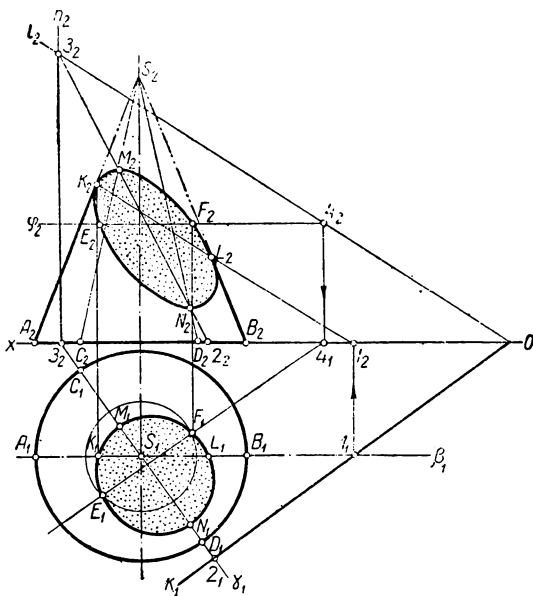


Fig. 315



the frontal with the limiting elements points  $K$  and  $L$ , in which the ellipse touches these elements, are found.

We then pass a plane  $\gamma$  perpendicular to the cutting plane  $\alpha$  through the axis of the cone. The trace  $\gamma_1$  is perpendicular to  $k_1$ . The plane  $\gamma$  intersects the cone along the generatrices  $SC$  and  $SD$ . It intersects plane  $\alpha$  along the line 2-3, which is a line of maximum

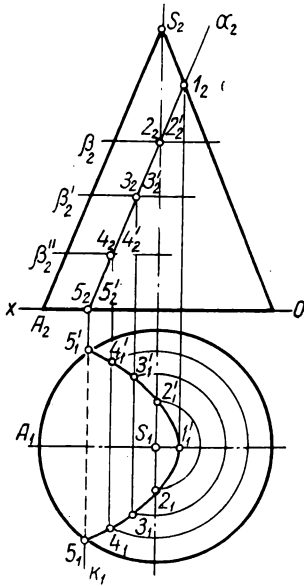


Fig. 316

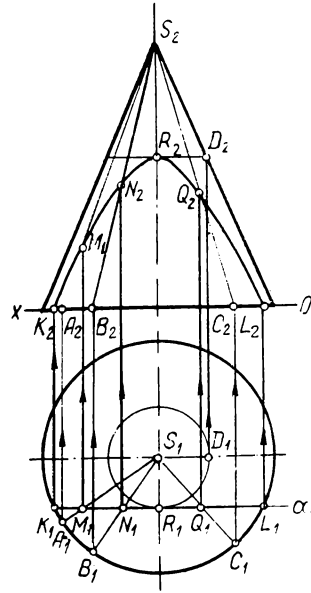


Fig. 317

inclination of plane  $\alpha$  to plane  $\Pi_1$ . This line contains points  $M$  and  $N$  in which the generatrices  $SC$  and  $SD$  of the cone will respectively intersect plane  $\alpha$ . In this case point  $M$  is the highest point and  $N$  the lowest point, a point of the ellipse given by the intersection of the cone by plane  $\alpha$ .

A series of other points of the ellipse may be constructed either with the help of the horizontal-projecting planes passed through the vertex  $S$ , or by means of horizontal planes. The points  $E$  and  $F$  of the ellipse are obtained by drawing the auxiliary horizontal plane  $\varphi$ . Plane  $\varphi$  intersects plane  $\alpha$  in a horizontal. The corresponding section of the cone is a circle, the diameter of which is equal to the length of the part of trace  $\varphi_2$  contained between the frontal projections  $S_2A_2$  and  $S_2B_2$  of the limiting elements of the cone. By describing this circle, with  $S_1$  as the centre, we obtain points  $E_1$  and  $F_1$  at its intersection with the horizontal projection of the horizontal of plane  $\alpha$ . Points  $E_2$  and  $F_2$  on trace  $\varphi_2$  of plane  $\varphi$  are determined with the help of projectors.

Joining points  $K_1, M_1, F_1, L_1, N_1, E_1$  and  $K_1$  by a curve we obtain the ellipse which is the horizontal projection of the intersection of the cone by plane  $\alpha$ . The frontal projection of this intersection is also an ellipse, passing through points  $K_2, M_2, F_2, L_2, N_2, E_2$  and  $K_2$ .

5. A cone intersected by a frontal-projecting plane  $\alpha$ , parallel to the limiting element  $SA$  of the cone, is shown in Fig. 316. In this case the plane section of the cone is a parabola. The frontal projection of the parabola coincides with trace-projection  $\alpha_2$  of the cutting plane.

To construct the horizontal projection of the parabola we draw a series of auxiliary horizontal planes ( $\beta, \beta', \beta''$ ), each of which intersects the cone in a circle and the plane  $\alpha$  in a horizontal perpendicular to plane  $\Pi_2$ . At the intersection of the horizontal projections of the horizontals with the horizontal projections of the respective circles, we obtain the points  $2_1, 2'_1, 3_1, 3'_1, 4_1, 4'_1$ . Points  $5'_1$  and  $5_1$  are the horizontal projections of points  $5$  and  $5'$  of the parabola and are contained in the circle of the base of the cone. The horizontal projection  $1_1$  of the highest point of the parabola, as well as points  $5'_1$  and  $5_1$ , are obtained by drawing projectors from points  $1_2$  and  $5_2$ .

By drawing a curve through points  $5_1, 4_1, 3_1, 2_1, 1_1, 2'_1, 3'_1, 4'_1$  and  $5'_1$  we obtain the horizontal projection of the parabola. The line  $5_15'_1$ , shown by dashes, is the horizontal projection of the line along which plane  $\alpha$  cuts the plane of the cone's base.

6. The intersection of a cone by a frontal plane  $\alpha$  which does not pass through the cone vertex  $S$  is shown in Fig. 317. In this case the section of the cone will be a hyperbola which is projected onto plane  $\Pi_1$  as a straight line coinciding with the trace  $\alpha_1$  and onto plane  $\Pi_2$  in its true size.

The points  $K$  and  $L$  of the hyperbola, in which it intersects plane  $\Pi_1$ , are determined by intersection of the base circle with trace  $\alpha_1$  of the cutting plane  $\alpha$ . The frontal projections  $K_2$  and  $L_2$  of these points are contained in ground line  $Ox$ .

To construct the frontal projection  $R_2$  of the special point  $R$ , the vertex of the hyperbola, with point  $S$  as the centre, we draw a circle whose radius is equal to the distance from point  $S_1$  to trace  $\alpha_1$ . This circle is the horizontal projection of the section of the cone given by a horizontal plane passing through point  $R$ . To find the frontal projection of this circle through point  $D_1$  we draw a projector to intersect the frontal projection of the right-hand cone generatrix in point  $D_2$ . A line is then drawn through point  $D_2$  parallel to ground line  $Ox$  and that part of it, which lies between the frontal projections of the limiting elements, is the projection on plane  $\Pi_2$  of the auxiliary circle of radius  $S_1D_1$ . Point  $R_2$  is the midpoint of this line.

7. The construction of the horizontal projection of a sphere cut by a horizontal-projecting plane  $\alpha$  (Fig. 318) is begun by finding

the axis of the ellipse, which is the circle of the section projected onto plane  $\Pi_1$ . The minor axis  $A_1B_1$  of the ellipse coincides with the horizontal projection of the major meridian of the sphere, the extremes  $A_1$  and  $B_1$  of this axis being determined by points  $A_2$  and  $B_2$  in which the frontal trace  $\alpha_2$  of the cutting plane intersects the frontal projection of the major meridian of the sphere. The projection  $E_2D_2$  of the major axis of the ellipse of the section on plane  $\Pi_2$  is the midpoint of line  $A_2B_2$ . Let us draw the auxiliary horizontal plane  $\beta$  so that its frontal trace  $\beta_2$  passes through point  $E_2 \equiv D_2$ . Having determined radius  $r$  of the circle given by the intersection of plane  $\beta$  and the sphere, with point  $C_1$  as the centre we describe a circle with this radius to intersect the projector drawn through point  $E_2 \equiv D_2$  in points  $E_1$  and  $D_1$ . Line  $E_1D_1$

Let us draw a projector through point  $M_2 \equiv N_2$ , i. e., through the intersection of trace  $\alpha_2$  of the cutting plane with the frontal projection of the equator of the sphere. This line will cut the circle, which is the horizontal projection of the equator of the sphere, in points  $M_1$  and  $N_1$ . It follows that, in points  $M$  and  $N$ , the circle given by the intersection of the sphere by plane  $\alpha$  touches the equator.

8. It should be noted that when constructing projections of plane sections, also in other cases, as few auxiliary lines as possible should be used. The drawing is simplified by using as

auxiliary lines any of the projections of the elements which happen to lie in a convenient position for this purpose.

An example is given in Fig. 319. This is the construction of the intersection of an elliptic cylinder by an oblique plane  $\alpha$ . The drawing in this case is simplified because all the auxiliary frontal-projecting planes, which are passed through the generatrices of the cylinder to find the points of their intersection with plane  $\alpha$ , are parallel.

So first the line of intersection of one such auxiliary plane, for example  $\beta$ , with plane  $\alpha$  is found. This is done by finding points 1 and 2 of the intersection of the corresponding traces of planes  $\alpha$  and  $\beta$ . The lines of intersection of other auxiliary frontal-projecting planes,  $\gamma$ ,  $\delta$  and  $\varphi$ , with plane  $\alpha$  will then be parallel to line 1-2 and, when constructing these lines, it is necessary to determine only one of their points in each case.

9. In engineering practice constructions are widely used in which planes are made to cut curved

surfaces. For example, in Fig. 320 the drawing of a connecting rod head is shown, and the way in which the elements of its surface flare into each other is shown by the lines of intersection of a complex body of revolution by two frontal planes,  $\alpha$  and  $\beta$ .

Let us subdivide the given surface into components of simple geometrical form, i. e., a sphere, a torus and a cylinder. The spherical surface of radius  $r_1$  is intersected by planes  $\alpha$  and  $\beta$  in arcs of circles projected onto plane  $\Pi_2$  without distortion. These planes cut the intermediate cylinder along generatrices.

The horizontal trace of the frontal plane  $\alpha$  intersects the horizontal projection of the equator of the sphere in point  $A_1$ . The

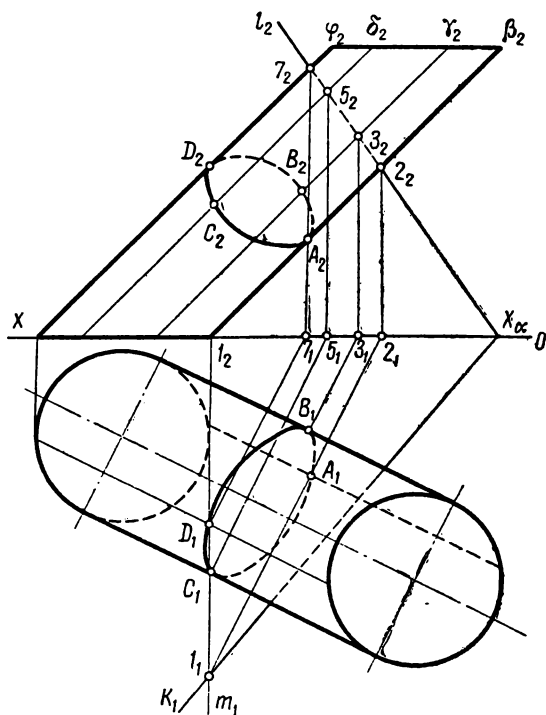


Fig. 319



Having built the projection of the cone on plane  $\Pi_4$ , we may find the part  $M_4K_4N_4$  of trace  $\alpha_4$ . This is the projection of the required section on plane  $\Pi_4$ . By drawing a series of generatrices of the cone we find first the horizontal, and then the frontal projections of the points of the section, including points  $M$ ,  $K$  and  $N$  in the  $\Pi_1$ - $\Pi_2$  system of planes of projection. After this, through the frontal projections of the points obtained an ellipse is drawn. This is the projection on plane  $\Pi_2$  of the section of the cone cut by plane  $\alpha$ . Through the horizontal projections of these points another ellipse, the projection of the section on plane  $\Pi_1$ , is drawn.

2. The multiview drawing (Fig. 322) shows the construction of the section of a sphere given by an oblique plane  $\alpha$ . The method of transformation of projections is also used in this case.

We project the sphere and plane  $\alpha$  onto a new plane of projections  $\Pi_4$  perpendicular both to plane  $\Pi_1$  and plane  $\alpha$ . The ground line  $s_{14}$  of the system of planes of projection  $\Pi_1$ - $\Pi_4$  is drawn perpendicularly to trace  $k_1$  of plane  $\alpha$ . The sphere is projected onto plane  $\Pi_4$  as a circle with its centre at point  $C_4$  which lies at the same distance from ground line  $s_{14}$  as the point  $C_2$  from ground line  $Ox$ .

The plane  $\alpha$  is projected onto plane  $\Pi_4$  as the line  $\alpha_4$ , and the section of the sphere as the line  $A_4B_4$ . This line coincides with line  $\alpha_4$ , the trace-projection of the plane on plane  $\Pi_4$ .

Line  $A_4B_4$  determines the length of the major axes of the ellipses in which the section of the sphere, a circle, is projected onto planes  $\Pi_1$  and  $\Pi_2$ . The projection of the centre  $C'$  of the circle of the section on plane  $\Pi_4$  is point  $C_4$ , the midpoint of line  $A_4B_4$ . We find the projection of this point  $C'$  on planes  $\Pi_1$  and  $\Pi_2$ . The projection  $C'_1$  of this point on plane  $\Pi_1$  is given by the intersection of the projector drawn through  $C_4$  perpendicular to  $s_{14}$  and the line passing through point  $C_1$  parallel to ground line  $s_{14}$ . The projection  $C'_2$  of point  $C'$  lies on the projector from  $C'_1$  perpendicular to ground

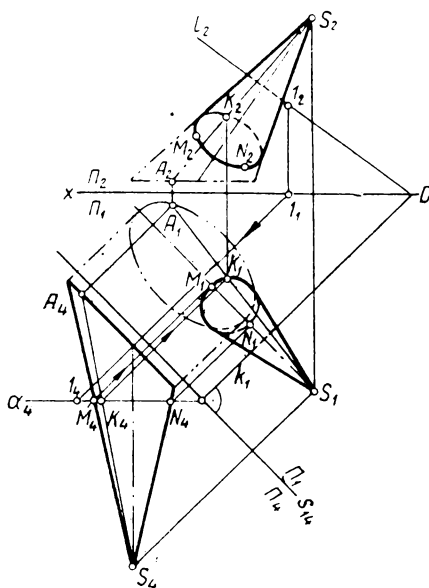


Fig. 321

line  $Ox$ , at a distance  $z_{C'}$  from that axis. This distance  $z_{C'}$  is obtained from the projection on plane  $\Pi_4$ .

Having found points  $C'_1$  and  $C'_2$  through them we draw lines respectively parallel to traces  $k_1$  and  $l_2$  and lay off on them to either side of points  $C'_1$  and  $C'_2$  distances equal to one half of the length of line  $A_4B_4$ . The lines  $D_1E_1$  and  $S_2F_2$  obtained are the

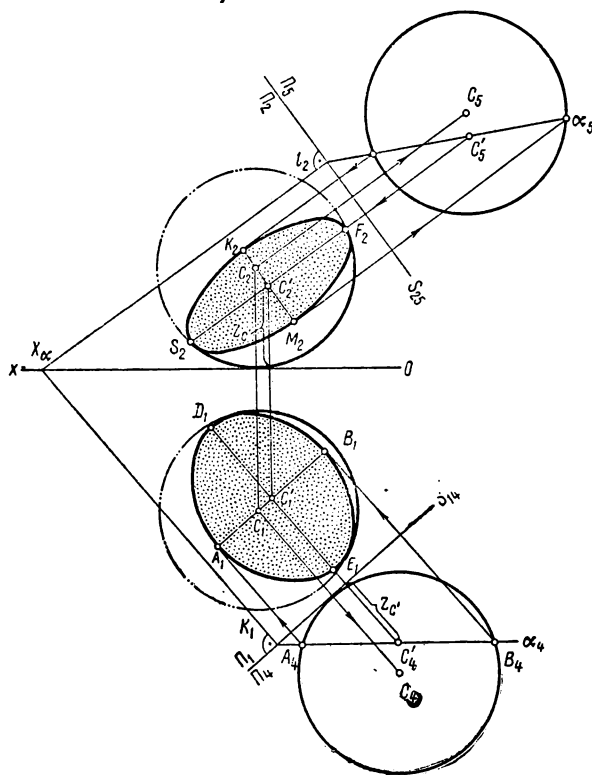


Fig. 322

major axes of the ellipses which are projections of the circular section on planes  $\Pi_1$  and  $\Pi_2$ .

The minor axis  $A_1B_1$  of the ellipse which is the projection of the circular section on plane  $\Pi_1$  is given by the projection of line  $AB$  on  $\Pi_1$ . To obtain the minor axis through points  $A_4$  and  $B_4$  draw projectors perpendicular to ground line  $s_{14}$  to intersect the line passing through point  $C'_1$  parallel to ground line  $s_{14}$ .

The length of the minor axis  $K_2M_2$  of the ellipse on plane  $\Pi_2$  is obtained by a similar transformation of projections, however in this case, a new plane  $\Pi_5$ , perpendicular both to plane  $\Pi_2$  and the cutting plane  $\alpha$ , is substituted for plane  $\Pi_1$ .

Having found the position and length of the axes of the ellipses which are projections of the circular section of the sphere on planes  $\Pi_1$  and  $\Pi_2$ , we can proceed to draw in the curves themselves.

3. We shall now examine the method of constructing plane sections by means of oblique projection. This type of projection has

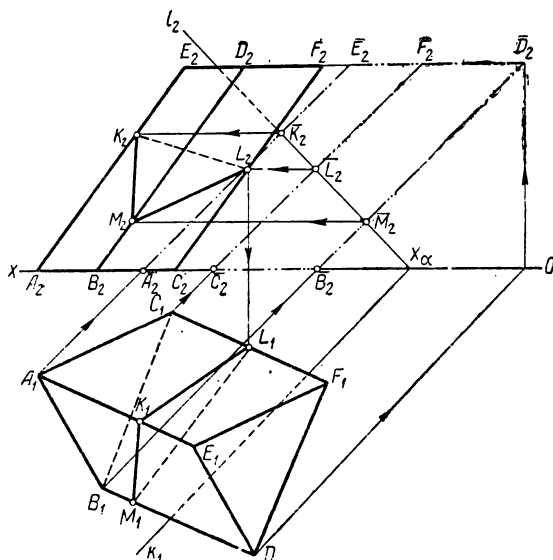


Fig. 323

already been used to determine the piercing point of a line with a plane (see Fig. 170).

The construction of the section of an oblique triangular prism given by an oblique plane  $\alpha$  is shown in Fig. 323.

The plane  $\alpha$  and prism are projected onto plane  $\Pi_2$  in a direction parallel to trace  $k_1$ . The projection of plane  $\alpha$  on plane  $\Pi_2$  is then line  $l_2$ , i. e., the frontal trace of plane  $\alpha$ . The prism is projected onto plane  $\Pi_2$  as  $\bar{A}_2\bar{B}_2\bar{C}_2\bar{E}_2\bar{F}_2\bar{D}_2$ . In Fig. 323 this projection is shown by dashes and two dots. The intersection of the oblique projections of the lateral edges of the prism with the trace  $l_2$ , points  $\bar{K}_2$ ,  $\bar{L}_2$  and  $\bar{M}_2$  determine the vertices of the triangle  $KLM$  received when the plane  $\alpha$  intersects the prism. Reverse projection gives the frontal projections  $K_2$ ,  $L_2$  and  $M_2$  of the vertices of the intersection triangle  $KLM$  and the horizontal projections  $K_1$ ,  $L_1$  and  $M_1$ .

4. The construction shown in Fig. 324 is carried out by using the related conformity properties of triangle  $KLM$  in which the surface of the pyramid  $SABC$  is intersected by plane  $\alpha$ .



In § 31 it was already noted that there exists a relationship between the projection of a plane figure and the figure itself. This is called the related conformity and this conformity exists between any two projections of the figure.

Therefore, problems which are solved by conventional methods of descriptive geometry may also be solved by using the method

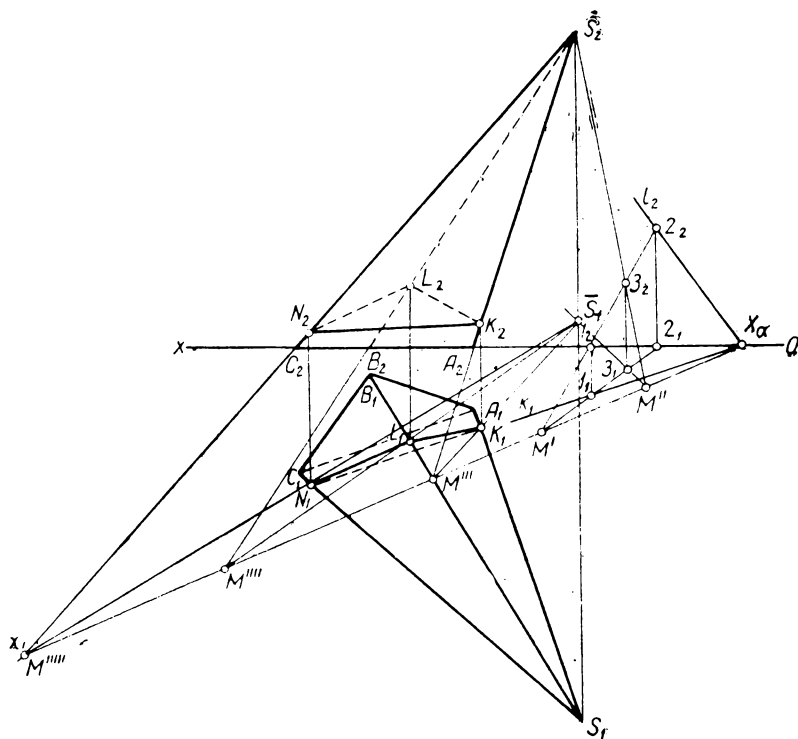


Fig. 324

of related conformity for any two projections of the projected objects.

Tackling this problem (Fig. 324), we must establish the related conformity between the section in plane  $\alpha$  and the horizontal projection of that same section of the pyramid. We begin by determining the axis and direction of relativity. To determine the direction of the relativity it is necessary to find one pair of related points.

The first duplicate point through which the axis of relativity will pass  $X_\alpha$ , is the point of convergence of traces of the plane. To determine the second duplicate point we draw in plane  $\alpha$  any line  $1-2$ , the projections of which,  $1_12_1$  and  $1_22_2$ , intersect in point  $M'$  which corresponds to itself. After this we pass the axis of relativity

through points  $X_\alpha$  and  $M'$ . The direction of relativity, in this particular case of related conformity is perpendicular to ground line  $Ox$ .

Point  $\bar{S}_1$ , related to point  $S_2$ , is now found by joining point  $S_2$  by a straight line with projection  $\beta_2$  of a point  $\beta$  taken at random on line 1-2. Line  $S_2\beta_2$  is produced to intersect the axis of relativity in point  $M''$ . Then by drawing the straight line  $M''\beta_1$ , which is related to line  $S_2\beta_2$ , to intersect with the line  $S_2S_1$ , which runs in the direction of relativity, we find point  $\bar{S}_1$  corresponding to the related point  $S_2$ .

Then we extend the projections  $S_2A_2$ ,  $S_2B_2$  and  $S_2C_2$  of the lateral edges of the pyramid to cut the axis of relativity in points  $M'''$ ,  $M''''$  and  $M'''''$ . By joining up these points with point  $\bar{S}_1$  we obtain lines  $\bar{S}_1M'''$ ,  $\bar{S}_1M''''$  and  $\bar{S}_1M'''''$  which are related to lines  $S_2A_2$ ,  $S_2B_2$  and  $S_2C_2$ .

The intersections of lines  $\bar{S}_1M'''$ ,  $\bar{S}_1M''''$  and  $\bar{S}_1M'''''$  with the horizontal projections  $S_1A_1$ ,  $S_1B_1$  and  $S_1C_1$  of the lateral edges of the pyramid give the horizontal projections  $K_1$ ,  $L_1$  and  $N_1$  of the vertices of triangle  $KLN$  which is the figure received when the pyramid  $SABC$  is cut by plane  $\alpha$ .

The frontal projection  $K_2L_2N_2$  of this triangle is constructed in the usual way with the help of projectors.

### Problems and Exercises

1. Which planes are generally used as auxiliary planes when constructing plane sections?

2. Describe the sequence of the constructions used to find the section of a polyhedron given by an oblique plane?

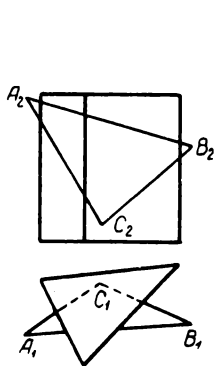


Fig. 325

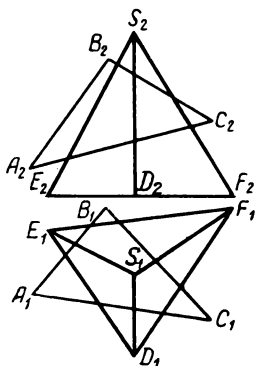


Fig. 326

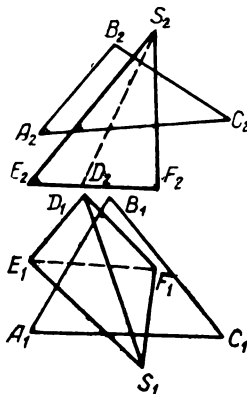


Fig. 327

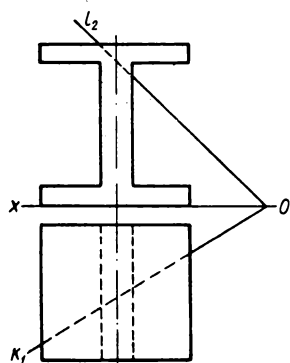


Fig. 328

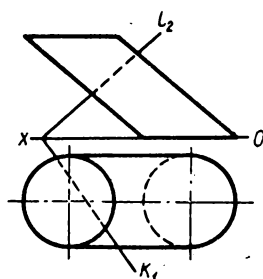


Fig. 329

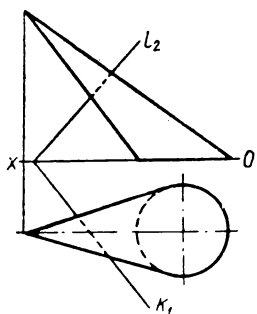


Fig. 330

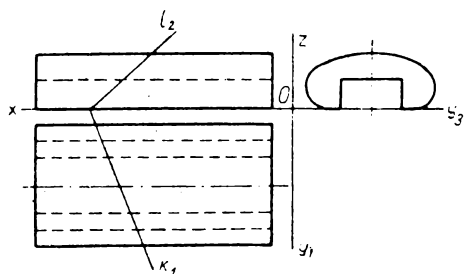


Fig. 331

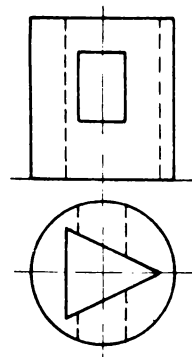


Fig. 332

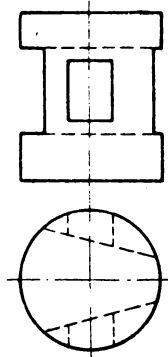


Fig. 333

3. Which points are called special points when constructing a plane section of a curved surface?

4. Construct the section given by an oblique plane when it cuts a triangular right prism. The traces of the plane make angles  $45^\circ$  and  $30^\circ$  with the ground line  $Ox$ . •

5. Construct the intersection of a triangular plate  $ABC$  with a prism. Show the visible parts of the plate by hatching (Fig. 325).

6. Construct the intersection of a triangular plate  $ABC$  with a pyramid (Figs. 326 and 327).

Hint. This problem should be solved by two methods, first, by using an auxiliary plane, and then by substituting planes of projection (see Fig. 321).

7. Construct the intersection of a given space figure by plane  $\alpha$  (Fig. 328).

8. Construct the intersection of a cylinder by a plane  $\alpha$  (Fig. 329). First, solve the problem in the conventional auxiliary planes, and then, by using oblique projection.

9. Construct the intersection of a cone by a plane  $\alpha$  (Fig. 330). First, solve the problem conventionally by passing auxiliary planes, then, employ the method of related conformity.

10. Construct the section of the handrail, shown in Fig. 331, when it is cut by plane  $\alpha$ . Find the true view of the section.

11. Construct the third projection of the cylinders with through holes, shown in Figs. 332 and 333.

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## Chapter IX

### INTERSECTION OF LINES AND SURFACES

#### § 37. Construction of Points of Intersection of Lines and Surfaces with the Help of Projecting Planes

1. In general it may be said that the construction of the point of intersection of a straight line with a surface consists of three operations, these are as follows: 1) an auxiliary surface is passed through the line; 2) the section cut by the auxiliary plane is found; 3) the point, or points, of intersection of the given line with the section which has been constructed is determined. Such points are the points of *intersection* of the line with the surface, i. e., the *piercing points*. The number of piercing points of a line depends on the form of the surface. A line generally pierces the surfaces of simple geometrical bodies, such as prisms, cylinders, sphere, etc., in two points, one of which may be called the *point of entry*, and the other, the *point of exit*.

If the surface which the line pierces is perpendicular to one of the planes of projection, then the piercing points may be determined very easily. For example, points  $K$  and  $L$ , in which line  $DE$  pierces

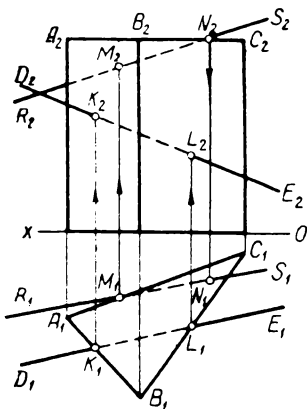


Fig. 334

the right triangular prism  $ABC$  (Fig. 334), are projected onto plane  $\Pi_1$  as points  $K_1$  and  $L_1$  in which the projection  $D_1E_1$  of line  $DE$  intersects the triangular horizontal projection of the two ends of the prism. Once points  $K_1$  and  $L_1$  are known, projectors are used to find the frontal projections  $K_2$  and  $L_2$  of points  $K$  and  $L$ .

The construction of points  $M$  and  $N$ , where another line, line  $RS$ , pierces the surface of the prism, is shown in Fig. 334. In this case point  $M$  lies in the hidden side of the prism, and point  $N$  lies in its upper face. The sequence of the construction is indicated by arrows.

The intersection points of a line and a right cylinder, the axis of which is perpendicular to plane of projection  $\Pi_1$ , may also be found without auxiliary constructions (Fig. 335).

It should be noted that when surfaces such as right prisms or right cylinders are intersected by curved lines, the piercing points are also found in this way. A space curve  $CD$  is shown as intersecting the surface of a cylinder in Fig. 335. The horizontal projections  $K'_1$  and  $L'_1$  of piercing points  $K'$  and  $L'$  are the points of

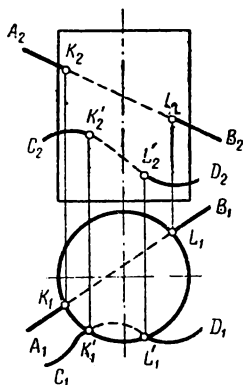


Fig. 335

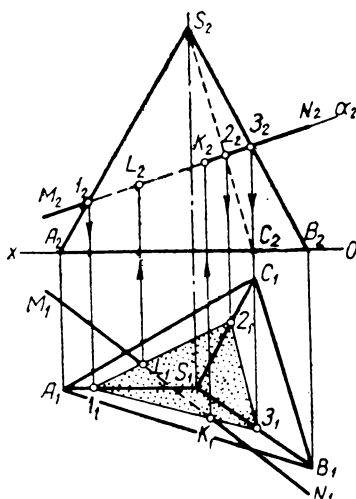


Fig. 336

intersection of curve  $C_1D_1$  with the circle which is the horizontal projection of the cylinder. The frontal projections  $K'_2$  and  $L'_2$  are determined by projectors drawn to meet the frontal projection  $C_2D_2$  of curve  $CD$ , as shown.

2. The construction of the points of intersection of line  $MN$  and the triangular pyramid  $SABC$  is shown in Fig. 336.

Draw an auxiliary frontal-projecting plane  $\alpha$  through line  $MN$ . The section of the pyramid cut by this plane is triangle  $1-2-3$ . First, points  $1_2$ ,  $2_2$  and  $3_2$ , where trace  $\alpha_2$  intersects frontal projections of the edges  $S_2A_2$ ,  $S_2C_2$  and  $S_2B_2$ , are found. Then, by means of these points and projectors, points  $1_1$ ,  $2_1$ ,  $3_1$  are determined. These points are joined by straight lines to obtain the triangle  $1_12_13_1$ , which is the horizontal projection of the section of the pyramid cut by the auxiliary plane  $\alpha$ .

The straight line  $MN$  contained in plane  $\alpha$ , together with triangle  $1-2-3$ , intersects the sides of this triangle in points  $K$  and  $L$  which are the points of intersection of this line with the surface of the pyramid. Points  $K_1$  and  $L_1$  are the points of intersections of

projection  $M_1N_1$  of line  $MN$  with the lines  $I_12_1$  and  $I_13_1$ , and the latter are projections of the sides of triangle  $I_12_13_1$ . By using projectors we find on  $M_2N_2$  the frontal projections  $K_2$  and  $L_2$  of the points  $K$  and  $L$  where line  $MN$  pierces the surface of pyramid.

When determining the visibility of separate parts of line  $MN$ , assuming this line is being projected onto planes  $\Pi_1$  and  $\Pi_2$ , it is necessary to take into account the visibility of the faces of the pyramid on these planes of projection. When projecting onto plane  $\Pi_1$  all three lateral faces are visible, so point  $K$ , contained in face  $SAB$ , and point  $L$ , contained in face  $SAC$ , are also visible. Therefore, the portions  $M_1L_1$  and  $K_1N_1$  of the horizontal projection of line  $MN$  are drawn as continuous lines, whereas the projection  $L_1K_1$  of the part of the line  $MN$  within the surface is shown by dashes. When projecting onto plane  $\Pi_2$  the face  $SAB$  is visible, but face  $SAC$  is not visible. Therefore the part  $K_2N_2$  of the frontal projection of line  $MN$  is shown as a continuous line. The portions  $I_2L_2$  and  $L_2K_2$  are shown by dashes, since they are the frontal projections of the part  $IL$ , which is hidden by the surface of the pyramid, and of the part  $LK$ , which is within the surface of the pyramid.

3. The points of intersection of a straight line  $a$  with a surface of revolution are also determined by passing a projecting plane through line  $a$  (Fig. 337).

The line in which the frontal-projecting plane  $\alpha$ , passing through line  $a$ , cuts the surface of revolution is a curve. The frontal projection of this curve coincides with trace  $a_2$ .

To construct the horizontal projection of the curve of intersection a series of auxiliary planes are drawn to cut the surface of revolution and plane  $\alpha$ . The sections received will be circles and horizontals perpendicular to plane  $\Pi_2$ . Having found the horizontal projections of these sections we determine points  $2_1$  and  $2'_1$ ,  $3_1$  and  $3'_1$ , in which horizontal projections of the circles and horizontals intersect. These points and the projections  $1_1$ ,  $4_1$  and  $4'_1$  are joined by a curve. The intersection of curve  $4_13_12_11_12'_13'_14'_1$  with projection  $a_1$  of line  $a$  gives points  $K_1$  and  $L_1$ . These are the horizontal projections of the points of intersection of line  $a$  with the surface of revolution. Having obtained points  $K_1$  and  $L_1$  we find the points  $K_2$  and  $L_2$  on the frontal projection  $a_2$  of line  $a$ . Finally, we determine the visibility of the separate parts of line  $a$ .

This solution demands the construction of an auxiliary curve. Although this involves extra work, it is unavoidable, since special constructions, not dealt with in this course, would have to be used to eliminate the construction of the curve.

4. The method of transforming projections to determine the points of intersection of a line  $AB$  and a sphere is shown in Fig. 338.

An auxiliary horizontal-projecting plane  $\alpha$  is passed through

line  $AB$ . The circular section of the sphere cut by this plane, together with line  $AB$ , is projected onto a new plane of projection  $\Pi_4$ . This plane  $\Pi_4$  is perpendicular to plane  $\Pi_1$  and parallel to plane  $\alpha$ . The centre  $C'$  of the circular section lies at the same altitude as the centre  $C$  of the sphere itself, therefore the projection  $C'_4$  of point  $C'$  on plane  $\Pi_4$  lies on the projector  $C_1C'_4$  at a distance from

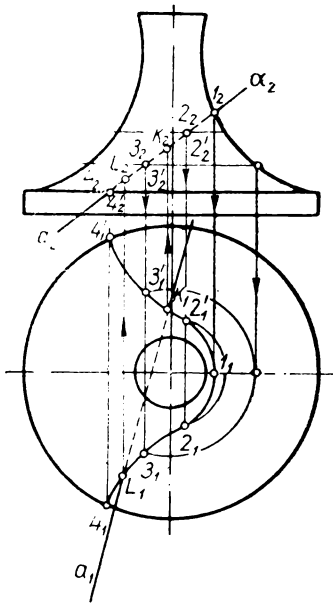


Fig. 337

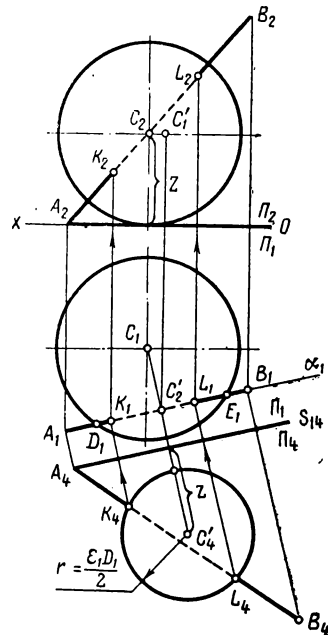


Fig. 338

the ground line  $s_{14}$  equal to the distance  $z$  for point  $C$ . A circle with a radius  $r$ , equal to one half of the length of line  $E_1D_1$ , is described with point  $C'_4$  as its centre. This circle intersects projection  $A_4B_4$  of line  $AB$  on plane  $\Pi_4$  in points  $K_4$  and  $L_4$ . These points are the projections on plane  $\Pi_4$  of points  $K$  and  $L$  in which line  $AB$  pierces the surface of the sphere.

Having obtained points  $K_4$  and  $L_4$  we first find the horizontal projections  $K_1$  and  $L_1$  and then the frontal projections  $K_2$  and  $L_2$  of points  $K$  and  $L$ . Point  $K$  is visible when projected onto plane  $\Pi_2$ , but it is hidden when projected onto plane  $\Pi_1$ . Point  $L$  is visible both when projected onto plane  $\Pi_1$  and onto plane  $\Pi_2$ . This is taken into account when determining the visibility of the separate parts of the projections of line  $AB$  on the multiview drawing. It is quite evident that the part  $KL$  of line  $AB$  is hidden both when projected onto plane  $\Pi_1$  and onto plane  $\Pi_2$  since it penetrates the sphere.



### § 38. The Use of Oblique Planes to Determine Points of Intersection of Lines and Surfaces

1. In many cases the construction of the points of intersection of a straight line and a surface is simplified by passing an oblique plane through the line, instead of a projecting plane.

For example, if a plane perpendicular to one of the planes of projection were to be used for the construction of the points of inter-

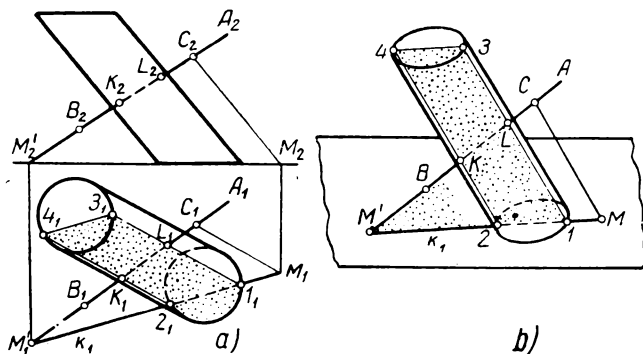


Fig. 339

section of line  $AB$  with the surface of an elliptical cylinder (Fig. 339), the section obtained would be an ellipse and one of its projections would have to be plotted by points. However, by passing an oblique plane parallel to the generatrices of the cylinder through line  $AB$  section of the cylinder will be given by two generatrices.

Such a plane may easily be constructed. Let us take any point  $C$  on line  $AB$  and through it draw line  $CM$  parallel to the generatrices of the cylinder. Then the plane determined by the intersecting lines  $AB$  and  $CM$  will be parallel to the generatrices of the cylinder.

The horizontal trace  $M$  of line  $CM$  and the horizontal trace  $M'$  of line  $AB$  are then found. The horizontal trace of the auxiliary plane determined by straight lines  $AB$  and  $CM$  passes through points  $M$  and  $M'$ . This trace  $k_1$  intersects the base of the cylinder contained in plane  $\Pi_1$  in points  $1$  and  $2$ . The horizontal projections of the generatrices  $1-3$  and  $2-4$ , along which the auxiliary plane cuts the lateral surface of the cylinder, are drawn through points  $1_1$  and  $2_1$ .

The points  $K_1$  and  $L_1$ , where lines  $2_1 4_1$  and  $1_1 3_1$  intersect line  $A_1 B_1$ , are the horizontal projections of the required points  $K$  and  $L$  where line  $AB$  pierces the surface of the cylinder. The frontal projections  $K_2$  and  $L_2$  of points  $K$  and  $L$  are found by drawing projectors through  $K_1$  and  $L_1$  to intersect the frontal projection  $A_2 B_2$  of line  $AB$ .

Planes, such as those considered, which reduce the construction lines to a minimum are the most simple oblique auxiliary planes.

2. A simple oblique auxiliary plane is also used in Fig. 340, *a* for constructing the points of intersection of line  $AB$  with the faces of an oblique prism, although in this case it would also have been possible to pass a projecting plane through line  $AB$ .

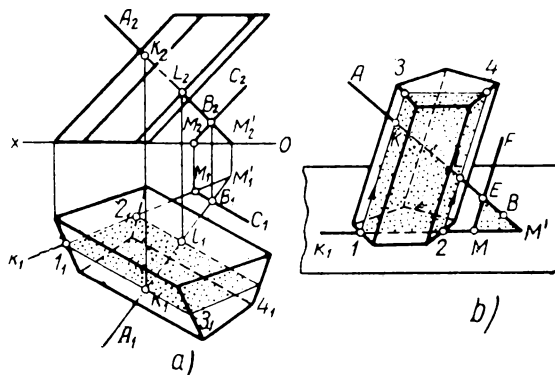


Fig. 340

As in the preceding construction, a line  $CM$  is drawn through point  $B$  on line  $AB$  parallel to the lateral edges of the prism. The plane  $\alpha$ , determined by the intersecting lines  $AB$  and  $CM$ , intersects the faces of the prism along lines  $1-3$  and  $2-4$  parallel to the lateral edges of the prism. The intersection of these lines with line  $AB$  gives the required points  $K$  and  $L$  where line  $AB$  pierces the prism.

This method of finding the points of intersection of a line and a pentagonal prism can be seen in axonometric projection in Fig. 340, *b*.

3. When determining the points of intersection of line  $AB$  and a cone (Fig. 341) the most simple oblique auxiliary plane will pass through line  $AB$  and the vertex  $S$  of the cone. Such an auxiliary plane intersects the lateral surface of the cone along two generatrices. To construct this plane we take on line  $AB$  points  $1$  and  $2$  at random and join them by straight lines to the vertex  $S$  of the cone. Having found the horizontal traces  $M$  and  $M'$  of lines  $S1$  and  $S2$  we pass the horizontal trace  $k_1$  of the plane through them and draw the generatrices  $SE$  and  $SD$  along which the plane intersects the surface of the cone. The points  $K$  and  $L$  in which generatrices  $SE$  and  $SD$  cut line  $AB$  are the points of intersection of this line and the cone.

4. When the base of the cone does not lie in the  $\Pi_1$  plane of projection, but in some frontal-projecting plane, as shown in

Fig. 342, the construction of the points of intersection of line  $AB$  and the cone remains almost unchanged. By drawing auxiliary lines  $S1$  and  $S2$  through the vertex  $S$  of the cone and points  $1$  and  $2$  taken on line  $AB$  we find the traces  $M$  and  $M'$  of these lines and

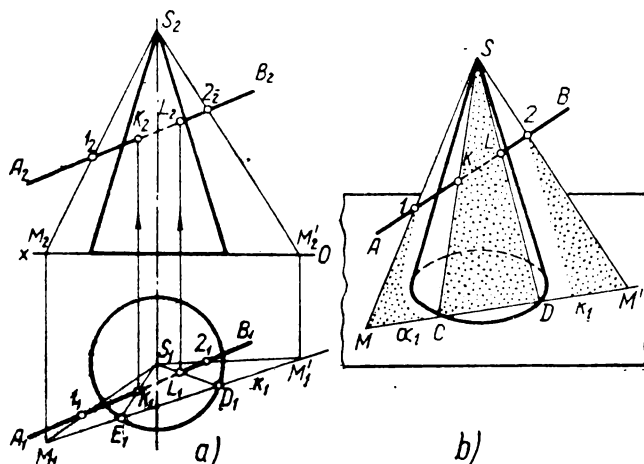


Fig. 341

trace  $MM'$  of the auxiliary plane on the plane in which the base of the cone lies. We obtain the points  $C$  and  $D$  at the intersection of trace  $MM'$  with the base of the cone. The generatrices  $SC$  and

$SD$ , along which the auxiliary plane intersects the surface of the cone, pass through these points. Points  $K$  and  $L$ , in which the line  $AB$  pierces the cone, lie at the intersection of the generatrices  $SC$  and  $SD$  with line  $AB$ .

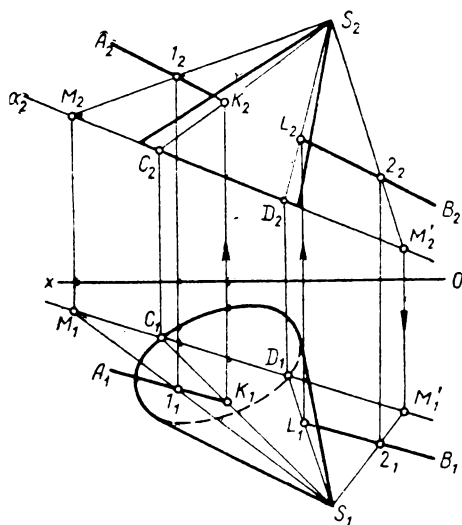


Fig. 342

5. Auxiliary planes which cut curved surfaces should be selected, whenever possible, so that the sections received are straight lines or circles and not curves in general, since the construction of the latter overloads the drawing with auxiliary lines and makes it less precise and legible.

Let us construct points  $K$  and  $L$  where line  $AB$  intersects a frustum of a cone by passing an auxiliary plane to intersect the cone along its generatrices (Fig. 343). Such a plane must pass through the vertex  $S$  of the cone. The projection of the vertex on plane  $\Pi_2$  will lie at a point obtained by extending the frontal projections of the limiting elements to intersect. However, in this case the frontal projection  $S_2$  of the vertex  $S$  lies outside the limits of the drawing.

In order to find the horizontal trace  $k_1 \equiv M_1M'_1$  of the auxiliary plane  $\alpha$  passing through line  $AB$  and the vertex  $S$  of the cone, it is necessary to obtain the horizontal trace  $M$  of line  $AB$  and the

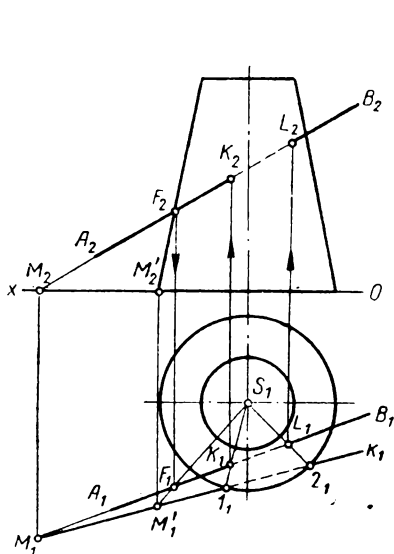


Fig. 343

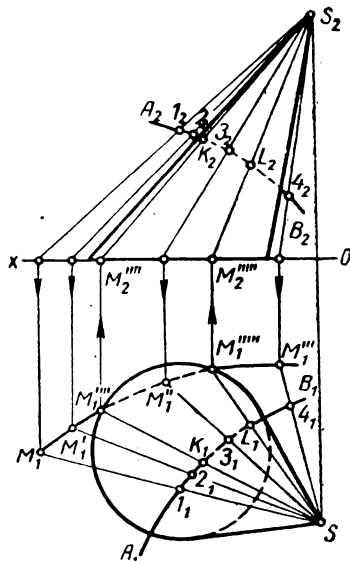


Fig. 344

horizontal trace of some other line contained in the auxiliary plane  $\alpha$ . For this purpose let us take line  $SF$  which passes through point  $F$  of line  $AB$  and through the vertex  $S$  of the cone. The frontal projection  $F_2$  of point  $F$  is taken as the intersection of projection  $A_2B_2$  and the frontal projection of the left-hand generatrix. Having found point  $F_2$  we determine point  $F_1$  on  $A_1B_1$ . Next the projection  $S_1F_1$  of line  $SF$  is drawn. The frontal projection of this line coincides with the frontal projection of the left-hand generatrix of the cone because both these projections pass through point  $S_2$ .

Using projections  $S_2F_2$  and  $S_1F_1$  we construct the horizontal trace  $M'$  of line  $SF$ . The horizontal trace  $k_1$  of the auxiliary plane  $\alpha$  is drawn through points  $M_1$  and  $M'_1$ . The horizontal projections  $S_1I_1$  and  $S_1J_1$  of the generatrices, along which the auxiliary plane cuts the surface of the cone, are then constructed. The points of

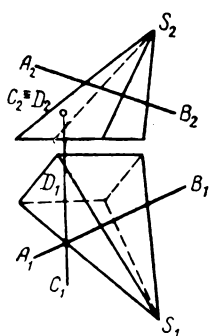


Fig. 345

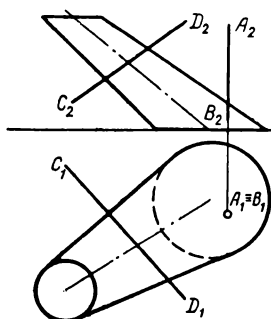


Fig. 346

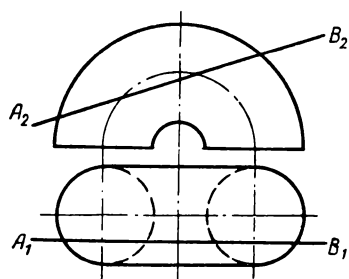


Fig. 347

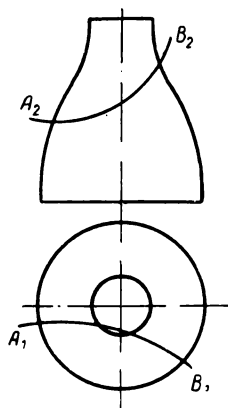


Fig. 348

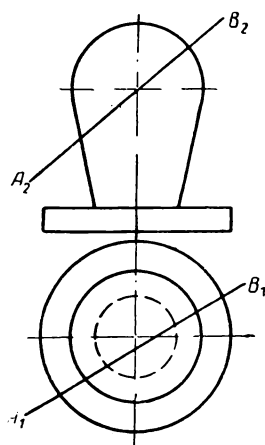


Fig. 349

intersection  $S_1I_1$  and  $S_12_1$  with  $A_1B_1$  determine the horizontal projections  $K_1$  and  $L_1$  of the required points  $K$  and  $L$  in which the line  $AB$  intersects the cone. The frontal projections  $K_2$  and  $L_2$  of points  $K$  and  $L$  are found on  $A_2B_2$  with the help of projectors.

6. The use of an auxiliary conical surface for determining the intersection points of a space curve  $AB$  with a cone is shown in Fig. 344. The auxiliary conical surface passed through curve  $AB$  and vertex  $S$  of the given cone intersects the latter along its generatrices.

First the horizontal trace of the auxiliary conical surface should be found. For this purpose straight lines are drawn through vertex  $S$  of cone and points 1, 2, 3, 4 of curve  $AB$ . Through the traces, or piercing points  $M_1, M'_1, M''_1$  and  $M'''_1$  of these lines a smooth curve is drawn. This curve intersects the circle of the base of the given cone in points  $M''''$  and  $M'''''$ .

The straight lines  $SM''''$  and  $SM'''''$ , drawn through the points obtained and the vertex of the cone, are the generatrices along which the two conical surfaces intersect. The points  $K$  and  $L$ , in which curve  $AB$  cuts these two generatrices, are the required points in which curve  $AB$  pierces the given cone.

#### Problems and Exercises

1. What types of auxiliary planes are used for determining the points of intersection of the surfaces of solid figures by straight lines?

2. What auxiliary surfaces are used for determining the points of intersection of surfaces of solid figures by curved lines?

3. What, in general, is the sequence in which the points of intersection of straight or curved lines with the surfaces of solid figures are constructed?

4. Find the points of intersection of lines  $AB$  and  $CD$  with the surface of the pyramid, shown in Fig. 345.

5. Find the points of intersection of lines  $AB$  and  $CD$  with the surface of the elliptical cone, shown in Fig. 346.

6. Find the points of intersection of line  $AB$  with the surface of the annular torus, shown in Fig. 347.

7. Find the points of intersection of curve  $AB$  with the surface of revolution, shown in Fig. 348.

Hint. Use an auxiliary cylindrical surface of intersection passing through curve  $AB$ .

8. Find the points of intersection of line  $AB$  with the surface of revolution, shown in Fig. 349.

## Chapter X

### INTERSECTION OF SURFACES

#### § 39. Construction of the Intersection of Surfaces Using Projecting Planes

1. The construction of points contained in lines in which surfaces intersect each other, also called *transition lines*, is carried out by introducing various auxiliary elements such as projecting planes, oblique planes and spherical surfaces. Transformation of projections and oblique projection are also used for this purpose. The choice of the auxiliary method to be employed is determined by the form and position of the intersecting surfaces.

The surfaces of two geometrical bodies may intersect each other in one or two closed lines. There is one line of intersection when the surfaces intersect only partially as if one surface “bites” into another, as it were.

Two lines of intersection are obtained when the intersection of the surfaces is complete, in other words, when one surface pierces right through the other. The construction of the lines of intersection of the surfaces of two polyhedra consists first in finding the points of intersection of the edges of one polyhedron with the faces of the other. Then the points of intersection of the edges of the second polyhedron with the faces of the first are determined. Finally, the points found are joined by straight lines, taking into account the visibility of the separate sides of the polygon in which the surfaces of the polyhedra intersect. In more complex cases special procedures facilitate the task of joining up the points contained in the line of intersection of surfaces of two polyhedra. One such method is given in § 40.

When determining the visibility of the portions of the lines of intersection of two surfaces the following rule should be kept in mind. If two lines are visible when projected onto some plane of projection, then their point of intersection is also visible when projected onto that same plane. The point of intersection of two lines not visible, or of one visible line with a hidden line, is not visible.

2. We shall now examine the case when a right regular hexagonal prism *I* is intersected by a triangular oblique prism *II* (Fig. 350).

Of the six lateral edges of prism *I* only edges *AB* and *CD* intersect the faces of prism *II*, since horizontal projections  $A_1B_1$  and  $C_1D_1$  of those edges lie within the limits of the horizontal projection of prism *II*.

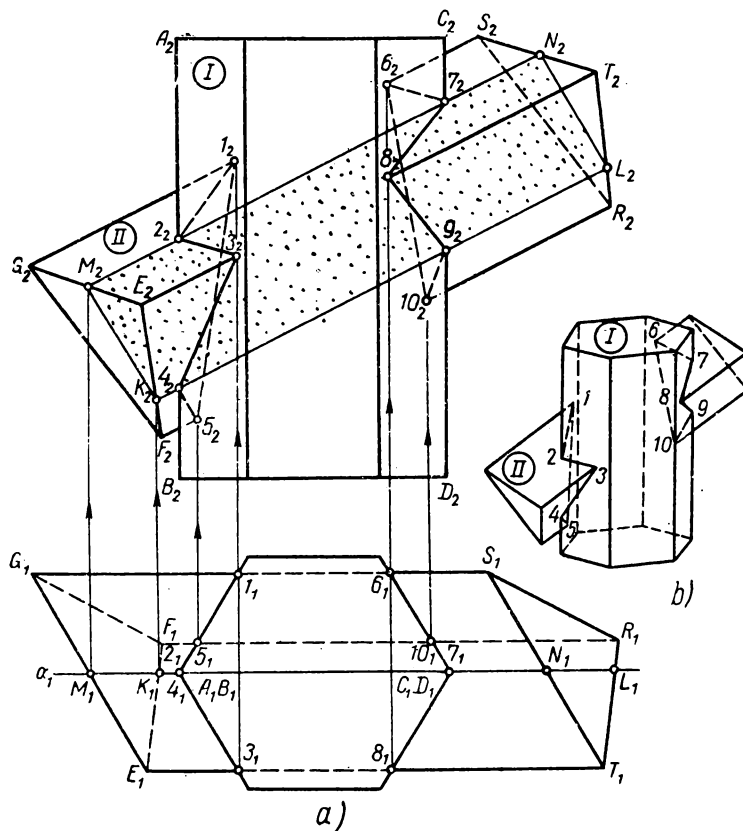


Fig. 350

First find the points of intersection of the lateral edges *AB* and *CD* of prism *I* with the faces of prism *II*. For this purpose pass an auxiliary frontal plane  $\alpha$  through the given edges. This plane intersects the face *EGST* of prism *II* in line *MN* and the face *EFRT* of the same prism in line *KL*. Since plane  $\alpha$  is parallel to the lateral edges of the prism *II*, these auxiliary lines *MN* and *KL* must also be parallel to those lateral edges. The section given when the surface of prism *II* is cut by plane  $\alpha$  is the quadrilateral *KLNM*.

The frontal projections  $M_2$  and  $K_2$  of points *M* and *K* are found



by means of the horizontal projections  $M_1$  and  $K_1$ . Then, by drawing lines  $M_2N_2$  and  $K_2L_2$  through  $M_2$  and  $K_2$  we find their intersection with projections  $A_2B_2$  and  $C_2D_2$  of edges  $AB$  and  $CD$ . These points of intersection are the frontal projections  $2_2, 4_2, 7_2$  and  $9_2$  of the points in which the edges  $AB$  and  $CD$  of prism  $I$  pierce the faces of prism  $II$ . The horizontal projections  $2_1, 4_1, 7_1$  and  $9_1$  of these points coincide with the horizontal projections of edges  $AB$  and  $CD$ , respectively.

The other points contained in the lines of intersection of the prisms, i. e., points  $1, 3, 5, 6, 8$  and  $10$ , can be obtained directly, without auxiliary constructions, since they are points in which the lateral edges  $ET, FR$  and  $GS$  of prism  $II$  intersect the lateral faces of prism  $I$ , which are horizontal-projecting planes. By passing projectors from the horizontal projections  $1_1, 3_1, 5_1, 6_1, 8_1$  and  $10_1$  we construct the frontal projections  $1_2, 3_2, 5_2, 6_2, 8_2$  and  $10_2$  of points  $1, 3, 5, 6, 8$  and  $10$ .

Points  $1_2, 2_2, 3_2, 4_2$  and  $5_2$ , and also points  $6_2, 7_2, 8_2$  and  $10_2$ , are joined by straight lines, taking into account their visibility when projecting onto plane  $\Pi_2$  the various sides of two pentagons in space in which prisms  $I$  and  $II$  intersect. The horizontal projections of the sides of these pentagons coincide with the horizontal traces of those lateral faces of prism  $I$  to which the sides belong.

The axonometric representation of these two intersecting prisms is seen in Fig. 350, *b*.

3. When constructing points of the lines of intersection of two curved surfaces with the help of projecting planes, both surfaces are intersected by a series of auxiliary planes. Sections of each surface given by one and the same plane are constructed. The intersection of these corresponding sections gives the required points. For instance, if a sphere and the circular cylinder intersecting it are cut by a horizontal plane  $\alpha$  (Fig. 351), then the intersection of the two circles obtained, which are sections of the sphere and the cylinder, gives points  $A$  and  $B$  contained in the lines of intersection of these surfaces.

The construction of the intersection of two cylinders by means of auxiliary frontal surfaces is shown in Fig. 352.

Since these planes are parallel to the axes of the cylinders, they cut the surfaces of the cylinders along generatrices. The points

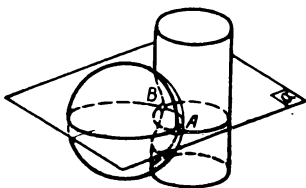


Fig. 351

$1$  and  $2$  where the front generatrix of cylinder  $II$  intersects cylinder  $I$ , and also points  $3$  and  $4$ , where the rear generatrix pierces cylinder  $I$ , are found directly. The horizontal projections  $1_1, 2_1, 3_1$  and  $4_1$  of these points are contained in the horizontal projection of the lateral surfaces of cylinder  $I$ . The frontal projections  $1_2, 2_2$ ,

$3_2$  and  $4_2$  are situated on the frontal projections of the front and rear generatrices of cylinder *II* and on the respective projectors.

Points *11* and *12* and also *9* and *10*, in which the top and bottom generatrices of cylinder *II*, respectively, intersect with the surface of cylinder *I*, are also determined directly. •

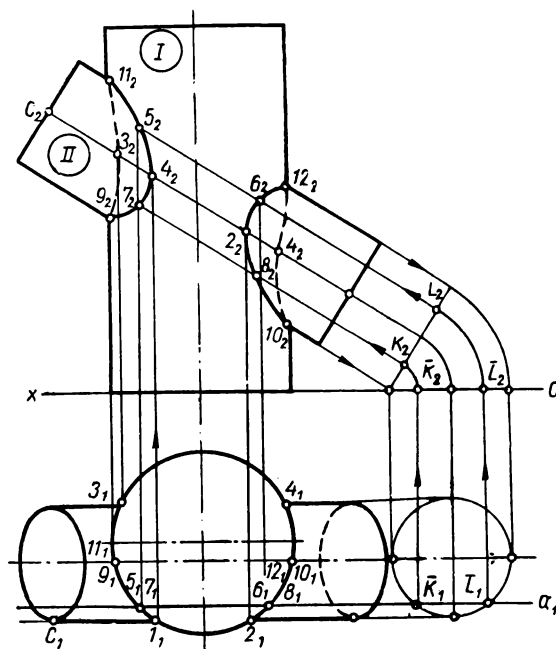


Fig. 352

In order to find points *5*, *6*, *7* and *8*, belonging to the lines of intersection of the surfaces of cylinders *I* and *II*, an auxiliary frontal plane  $\alpha$  is drawn. The generatrices along which plane  $\alpha$  cuts the surface of the two cylinders are constructed. In order to make the construction more convenient and exact the circular base of cylinder *II* is rotated to lie in plane  $\Pi_1$ . Having obtained points  $5_1$ ,  $7_1$  and  $6_1$ ,  $8_1$  where trace  $\alpha_1$  of plane  $\alpha$  intersects the horizontal projection of the lateral surface of cylinder *I*, we construct the frontal projection of the generatrix which contains points *5* and *7* and the generatrix which contains points *6* and *8*. Then the points  $\bar{K}_1$  and  $\bar{L}_1$ , where  $\alpha_1$  intersects the circular base of cylinder *II* made to coincide with plane  $\Pi_1$ , are found. Having done this we can obtain first points  $\bar{K}_2$  and  $\bar{L}_2$ , and then points  $K_2$  and  $L_2$ .

Through  $K_2$  and  $L_2$  the frontal projections of the generatrices along which plane  $\alpha$  cuts the surface of cylinder *II* are drawn and

extended to intersect the frontal projections of the corresponding generatrices in which this same plane cuts cylinder 1. This gives us the frontal projections of points 5, 6, 7 and 8 contained in the line of intersections of the cylinders. A series of auxiliary frontal planes similar to plane  $\alpha$  are drawn. This gives us further points belonging to the lines of intersection of the cylinders. Continuous curves are drawn through the points obtained. When drawing these

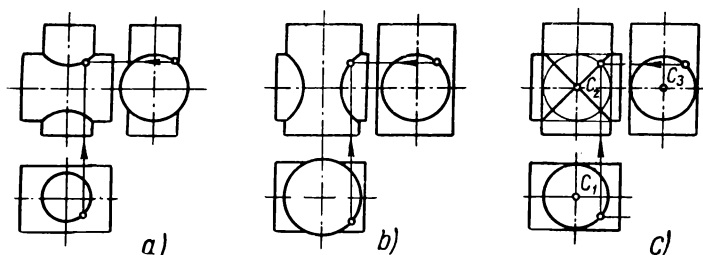


Fig. 353

lines the visibility of the various points projected onto plane  $\Pi_2$  should be taken into account.

4. Intersecting cylinders are shown in Fig. 353, *a*, *b*, *c*. The cylinders with horizontal generatrices in all three cases have the same diameter, but the cylinders with vertical generatrices have different diameters. Depending on the size of the diameters of cylinders with vertical generatrices the lines of intersection of the cylinders are projected onto plane  $\Pi_2$  either in the form of curves — hyperbolas (Fig. 353, *a*, *b*) — or as straight lines (Fig. 353, *c*). In the latter case both cylinders are of the same diameter. Such cylinders may be considered as surfaces described about a sphere with its centre in point  $C$  and its radius equal to the distance from point  $C$  to the generatrices of the cylinder. The two plane curves — ellipses — obtained by the intersection of the given cylinders have two points in common with the surface of the sphere about which the two cylindrical surfaces are described.

5. An example of the construction of the intersection of the two cylindrical surfaces making up a tee-pipe connection is shown in the multiview drawing (Fig. 354). The points used in the construction of this line of intersection, or transition, were found by using auxiliary horizontal planes  $\alpha$ ,  $\beta$  and  $\gamma$ .

6. The construction of the intersection of a cylinder and a cone, both axes of which are perpendicular to plane  $\Pi_1$  (Fig. 355), was carried out by using auxiliary horizontal planes  $\alpha$ ,  $\beta$  and  $\gamma$  to intersect the cone and cylinder in circles. These circles are projected

onto plane  $\Pi_2$  as straight lines and onto plane  $\Pi_1$  as circles. The horizontal projections of all the circles in which the planes intersect the surface of the cylinder coincide with the projection of the cylinder on plane  $\Pi_1$ . The diameters of the circular sections cut on the cone by the auxiliary planes are determined from the frontal projection, as indicated by an arrow for the circular section of the cone cut by plane  $\gamma$ .

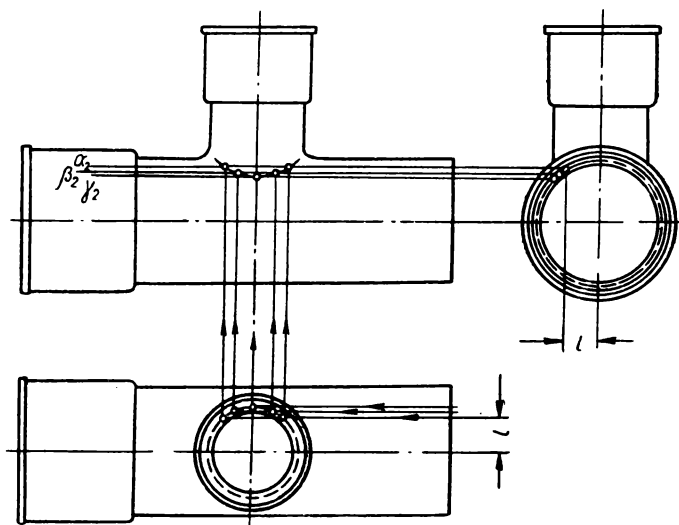


Fig. 354

The intersection of the circle, which is the projection of the cylinder on  $\Pi_1$ , with the horizontal projections of the circles, obtained when the cone is cut by planes  $\alpha$ ,  $\beta$  and  $\gamma$ , determine the horizontal projections  $A_1, B_1, C_1, D_1, E_1$  and  $F_1$  of points  $A, B, C, D, E$  and  $F$ , through which the line of intersection of the cylinder and cone pass. The frontal projections  $A_2, B_2, C_2, D_2, E_2$  and  $F_2$  of these points are obtained by passing projectors through  $A_1, B_1, \dots, F_1$  to the frontal traces of the corresponding planes.

The special points, the highest point  $M$  and lowest point  $N$ , are found with the help of plane  $\delta$  passed through the axes of the cylinder and cone. The points of intersection of the left-hand (L. H.) and right-hand (R. H.) generatrices of the cylinder with surface of the cone are constructed by passing planes  $\varphi$  and  $\xi$  through the axis of the cone and through each of the two generatrices of the cylinder. Then the generatrices along which these planes intersect the cone are found. This construction is shown for point  $K$ .

7. Three cases of intersection of a cylinder with a cone are shown in Fig. 356. The three cylinders are of the same diameter.

Comparing the diameter of the cylinder with that of the circular section of the cone whose centre coincides with point  $C$ , i. e., with the point of intersection of the axes of the two bodies, it is seen that in the first case (Fig. 356, *a*) the diameter of this section of the cone is smaller than the diameter of the cylinder, in the second case (Fig. 356, *b*) these diameters are equal, and in the third case

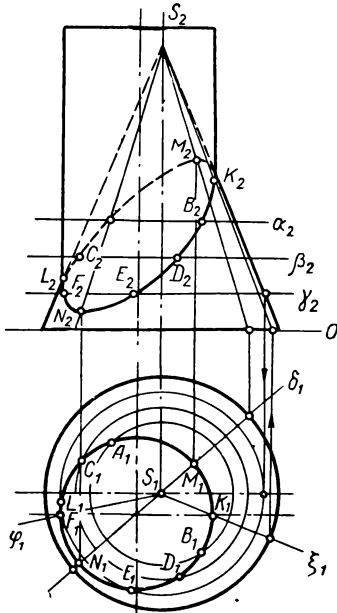


Fig. 355

(Fig. 356, *c*) the diameter of the circle of the cone is larger than that of the cylinder. The second case (Fig. 356, *b*) differs from the first and third in that a circle may be described from point  $C_2$  to touch both the outline generatrices of the cylinder and of the cone. Therefore, the surfaces of these two bodies are tangential to a sphere with its centre in point  $C_2$ . These two surfaces intersect each other in two plane curves, ellipses, having two points in common.

8. The construction of the lines of intersection of the frustum of a cone with a sphere is shown in the multiview drawing (Fig. 357).

Since the axis of the cone and the vertical axis of the sphere are located in the same frontal plane, the points  $A$  and  $B$  of intersection of the R. H. and L. H. generatrices of the cone with the principal meridian of the

sphere are determined directly. Points  $A_2$  and  $B_2$  are the frontal projections and  $A_1$  and  $B_1$  the horizontal projections of these points,  $A$  being the highest point and  $B$  the lowest point of the line of intersection of the cone and sphere.

In order to construct points  $D$  and  $E$ , in which the front and

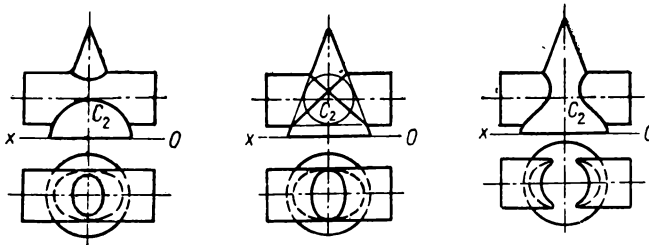


Fig. 356

rear generatrices of the cone intersect the sphere, profile plane  $\beta$  is drawn. This plane intersects the surface of the sphere in a circle with radius  $r'$ , which is projected onto plane  $\Pi_3$  in its true size. Therefore an arc of radius  $r'$  is described with point  $C_3$  as its centre. This arc intersects the profile projections of the front and rear generatrices of the cone in points  $D_3$  and  $E_3$ . These are the boundaries of visibility in the projection on plane  $\Pi_3$  of the various parts of the line of intersection. At the same time they are the points of intersection of the profile projections of the front and rear generatrices of the cone with the profile projection of the curve of intersection of the surfaces of the cone and sphere. Using points  $D_3$  and  $E_3$  we find points  $D_1$ ,  $E_1$ ,  $D_2 \equiv E_2$ .

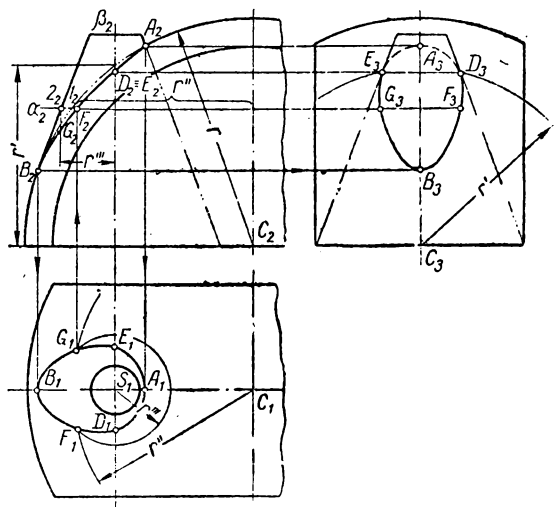


Fig. 357

The intermediate points  $F$  and  $G$  are constructed by drawing a horizontal plane  $\alpha$  to cut the sphere in a circle of a radius  $r''$  and to cut the cone in a circle of a radius  $r'''$ . These circles are projected onto plane  $\Pi_1$  as true views, and their intersection determines the horizontal projections  $F_1$  and  $G_1$  of points  $F$  and  $G$  contained simultaneously in plane  $\alpha$  and in the line of intersection of the cone and cylinder. Using points  $F_1$  and  $G_1$  the points  $F_2$  and  $G_2$  are found with the help of projectors drawn through  $F_1$  and  $G_1$  to intersect trace  $\alpha_2$  of the auxiliary horizontal plane  $\alpha$ .

The construction of the profile projections  $A_3$ ,  $B_3$ ,  $G_3$  and  $F_3$ , of points  $A$ ,  $B$ ,  $G$  and  $F$  is carried out as usual by constructing the third projection of the points using two of their known projections.

Using other auxiliary horizontal planes, a series of intermediate points of the curve of intersection of the given cone and sphere may be found.

When a sufficient number of points has been found the projections of the curve may be drawn through the corresponding projections of the points.

## § 40. Construction of the Intersection of Surfaces by Means of Oblique Planes

1. An oblique plane passing through a line parallel to the edges of a prism, intersects the surface of the prism along lines parallel to these edges, see Fig. 340, *b*. If we intersect two prisms by a plane  $\alpha$  which passes through two lines, one of which is parallel to the edges of the first prism and the other parallel to the edges of the second prism, we obtain lines respectively parallel to these edges on the surface of each prism.

These basic premises when selecting the most simple oblique section planes such as  $\alpha$ ,  $\alpha'$ ,  $\alpha''$ ,  $\alpha'''$  and  $\alpha''''$  for the construction of points contained in the lines of intersection of the two oblique prisms  $ABCDE$  (*I*) and  $KLMN$  (*II*), the bases of which are contained in plane  $\Pi_1$ , see Fig. 358.

Let us determine the direction of the horizontal traces  $f_1$ ,  $q_1$ ,  $p_1$ ,  $s_1$  and  $k_1$  of the auxiliary cutting planes which are parallel to the lateral edges of both prisms. Take any convenient point  $Q$  and draw two lines through it, one,  $QM$ , parallel to the lateral edges of the first prism, the other,  $QM'$ , parallel to the lateral edges of the second prism. Having found the horizontal traces  $M$  and  $M'$  of these lines draw the trace  $MM'$ . This line  $MM'$  determines the direction required.

The first auxiliary plane  $\alpha$  is passed through edge  $A$  of the first prism. The horizontal trace  $f_1$  of plane  $\alpha$  will pass through the horizontal trace ( $\bar{I} \equiv \bar{2}$ ) of this edge  $A$ . The trace  $s_1$  of the last auxiliary plane  $\alpha'''$  passes through the horizontal trace ( $\bar{9} \equiv \bar{10}$ ) of edge  $D$  of the prism. The other auxiliary planes lie between these two extreme planes.

Certain parts of the second prism, which lie outside the space contained between planes  $\alpha$  and  $\alpha'''$  do not intersect the surface of the first prism. The areas of the horizontal projection of the base of the second prism, which correspond to these parts, lie below trace  $f_1$  and above trace  $s_1$ . They are indicated on the drawing by hatching.

Plane  $\alpha$ , passed through the lateral edge  $A$  of the first prism, intersects the surface of the second prism along lines  $\bar{1}-1_1$  and  $\bar{2}-2_1$  parallel to its lateral edges. In Fig. 358 only the horizontal projections of these lines are shown. The horizontal projection of edge  $A$  of the first prism intersects the horizontal projections of lines  $\bar{1}-1_1$  and  $\bar{2}-2_1$  in points  $1_1$  and  $2_1$ , respectively. These are the horizontal projections of the points of intersection 1 and 2 of edge  $A$  of the first prism with the surface of the second prism.

By drawing projectors from points  $1_1$  and  $2_1$  perpendicular to ground line  $Ox$  to intersect the frontal projection of edge  $A$ , we

obtain the frontal projections  $1_2$  and  $2_2$  of points 1 and 2. In Fig. 358 the projectors are not shown.

2. The method of similar denotations. By a similar construction, with the help of plane  $\alpha'''$  passed through edge  $D$ , we

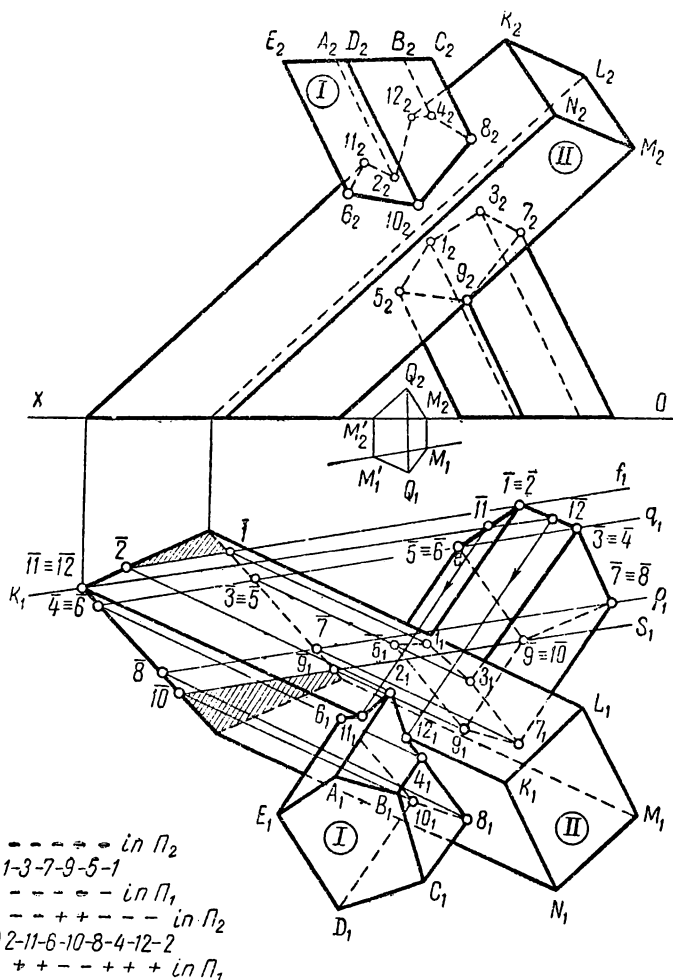


Fig. 358

obtain the points of intersection 9 and 10 of this edge with the surface of the second prism.

Plane  $\alpha''$ , the horizontal trace of which is  $q_1$ , passes through edges  $B$  and  $E$  of the first prism and intersects the surface of the second prism along the lines  $\bar{3}-3$  and  $\bar{4}-4$ . The intersection of these



lines  $\bar{3}-3$  and  $\bar{4}-4$  with edges  $B$  and  $E$  gives points  $3$  and  $4$  and also points  $5$  and  $6$  of the line of intersection of the two given prisms.

The planes  $\alpha'$  and  $\alpha''$ , passed through the edge  $C$  of the first prism and through the edge  $K$  of the second prism, determine two pairs of points,  $7$  and  $8$ , and  $11$  and  $12$ , belonging to the line of intersection of the prisms.

After obtaining twelve points, contained at one and the same time in the surfaces of both the first and second prisms, join up these points to give two space polygons. The sequence in which the points are connected by straight lines is established with the help of the method of similar notations.

In accordance with this method use projected points  $1, 2, 3, \dots, 12$  parallel to the edges of the prism onto the base of that prism (Fig. 358). The oblique projections of these points received on plane  $\Pi_1$  will coincide with the horizontal traces of the lateral edges and lines along which the auxiliary planes intersect the faces of the prism. The oblique projections of the points on plane  $\Pi_1$  should be denoted in the same way as the corresponding points of the lines of intersection of the prisms. For example, the oblique projection of point  $1$  on the base of the second prism is denoted as  $\bar{1}$  and the two coinciding projections of points  $1$  and  $2$  on the base of the first prism as  $\bar{1} \equiv 2$  and so on.

The sequence in which the oblique projections of these points are situated on the bases of the prisms corresponds to the sequence of the points on the lines of intersection in space. Therefore, we can determine the sequence in which the points should be joined in space by making the round of the oblique projections. The direction will be clockwise and we can start with any one of the points, say point  $1$ . So, beginning with points  $\bar{1}$  on the base of each prism, we note down on paper the number  $1$  and pass on to the nearest points. On the base of the first prism point  $\bar{12}$  lies next to point  $\bar{1}$  and on the base of the second prism we find the duplicate point  $\bar{3} \equiv \bar{5}$ . Having established that the notations of these points differ, we bypass point  $\bar{12}$  and take the next point  $\bar{3} \equiv \bar{4}$  on the base of the first prism. The points  $\bar{3} \equiv \bar{4}$  and  $\bar{3} \equiv \bar{5}$  have one common number  $\bar{3}$ . We then write down  $3$  next to  $1$ . Continuing the movement round the bases of both prisms we find a common notation of points  $\bar{7} \equiv \bar{8}$  and  $7$ , so we write  $7$  next to  $3$ . The next points with similar denotations are  $\bar{9} \equiv \bar{10}$  and  $9$ . Consequently, we note down  $9$  next to  $7$ .

Still continuing this movement around the base of the second prism we come to the bottom hatched part of the base. For this base the direction should now be reversed, i. e., we must move towards point  $\bar{3} \equiv \bar{5}$ , which corresponds to the closed outline of the lines of intersection in space. Moving in this new direction we establish the

similar notation  $\bar{5}$  in points  $\bar{5} \equiv \bar{6}$  and  $\bar{3} \equiv \bar{5}$ . Therefore, after 9 we write down 5. Finally, by passing  $\bar{1}\bar{1}$  we return to the initial points  $\bar{1}$  and  $\bar{1} \equiv \bar{2}$  and again write down the common notation 1. The line of intersection of the prisms is now closed. We have a list of numbers 1-3-7-9-5-1. The corresponding projections of points contained in the lower line of intersection of the prisms are joined up in this same order.

By carrying out a similar operation in a counter-clockwise direction we establish the sequence in which the points of the second, upper line of intersection of the prisms should be joined. This sequence is 2-11-6-10-8-4-12-2. The starting point in this case was point 2.

The visibility of the separate points, and the parts of the lines of intersection adjacent to them, is readily determined from the oblique projections of these points on the base of the prisms. If two oblique projections of a point contained in the lines of intersection, or transition line, have similar notations and are visible in the given plane of projections, then the orthographic projections of this point and the part of the transition line adjacent to it are also visible.

Similarly, if one or both oblique projections of one of the points are hidden, its orthographic projection and the projection of the parts of the line of intersection adjacent to this point on that same plane of projection are also hidden.

The visibility of separate parts of the line of intersection of the prisms can be indicated by the signs + or -, as seen in Fig. 358. The visible parts are denoted by the plus sign and the parts not visible by the minus sign.

The projections of the lower line of intersection of the prisms, which is a plane pentagon 1-3-7-9-5, are not visible. The projections of the various sides of the space heptagon 2-11-6-10-8-4-12, which is the upper line of intersection of the prism, are drawn either as continuous or broken lines, depending on the visibility of each of the sides when projected onto planes  $\Pi_1$  and  $\Pi_2$ .

3. The construction of the intersection of a tetragonal prism with a triangular pyramid is shown in Fig. 359. The bases of these two solid figures are not contained in any of the planes of projection.

In order to obtain points on the line of intersection of the prism and pyramid auxiliary planes are passed through line  $SM$  drawn through the vertex  $S$  of the pyramid parallel to the lateral edges of the prism. A similar plane was used when constructing points of intersection in Fig 341.

Since the bases of the prism and pyramid are not contained in the horizontal plane of projection an auxiliary horizontal plane  $\varphi$  is drawn to intersect the lateral surface of the pyramid in triangle 1-2-3 and the lateral surface of the prism in the tetragon 4-5-6-7.

The horizontal projections of these lines of intersection are shown by dashes and two dots. Point  $M \equiv M_1$  is the horizontal projection of the piercing point of line  $SM$  in plane  $\varphi$ . Through this point we draw the traces in plane  $\alpha$  of the auxiliary cutting planes passing through line  $SM$ . This sheath of planes opens like a fan in space. We then find the horizontal projection of the lines

along which these planes cut the surfaces of the prism and pyramid. On the surface of the prism these section lines are the lateral edges themselves, or lines parallel to them. On the surface of the pyramid the section lines are lateral edges or lines contained in the faces and passing through the vertex  $S$ . At the intersection of the horizontal projections of the respective section lines we find the horizontal projections  $8_1, 9_1, 10_1, 11_1, 12_1, 13_1, 14_1$  and  $15_1$  of points contained in the line of intersection of the prism and pyramid.

The frontal projections  $8_2, 9_2, 10_2, 11_2, 12_2, 13_2, 14_2$  and  $15_2$  of these points are found by passing projectors through their horizontal projections.

The sequence in which corresponding projections of the eight points obtained are joined up to form the two closed lines of intersection, 8-12-10 and 13-14-11-9-15, is determined by the

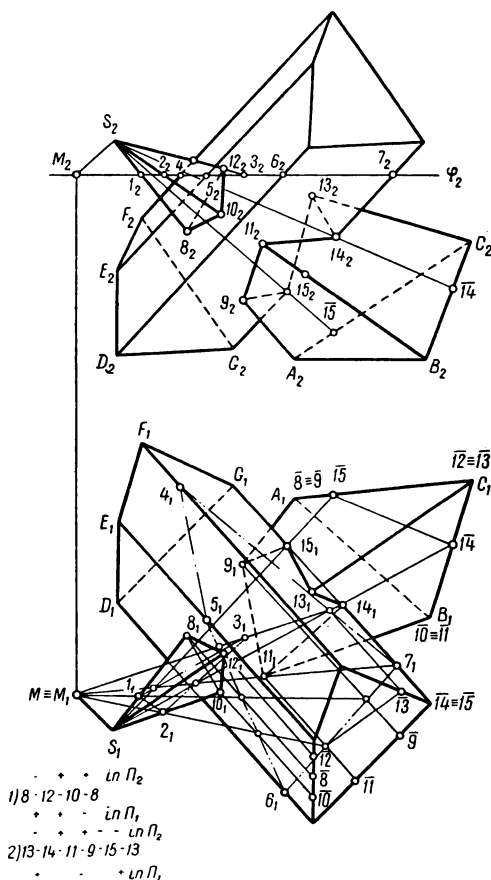


Fig. 359

method of similar denotations. For this purpose the points of intersection must be projected obliquely onto the bases of the two given bodies.

4. When constructing the line of intersection of the surfaces of two oblique cylinders the most simple auxiliary planes are those drawn parallel to the generatrices of the cylinders (Fig. 360, *a*).

When the surfaces of a cylinder and cone intersect, the simple auxiliary planes are those drawn through the vertex  $S$  of the cone parallel to the generatrices of cylinders (Fig. 360, *b*). Finally, to find the intersection of the surfaces of two cones the most convenient auxiliary planes are those which pass through the vertices of the cones (Fig. 360, *c*).

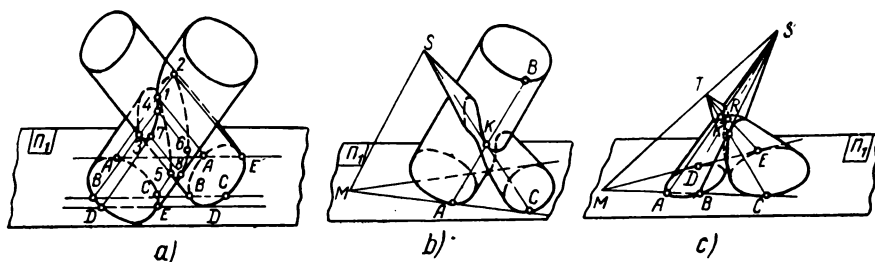


Fig. 360

Similar auxiliary planes are used when constructing the lines of intersections of an oblique cylinder and a prism, of a cylinder and a pyramid, of a cone and a prism, of a cone and a pyramid. In every case these auxiliary planes give us section lines on each of the bodies. The intersections of these section lines give us points contained in the lines of intersection of the two given surfaces.

5. An example of the use of simple auxiliary planes to determine the transition line of two curved surfaces is the construction of the intersection of the surfaces of the elliptical cylinders, shown in Fig. 361.

The direction of the horizontal traces of the auxiliary planes is determined as follows: Lines  $QM$  and  $QM'$  are drawn respectively through any point  $Q$  parallel to the generatrices of the given cylinders  $I$  and  $II$ . The horizontal traces of all the auxiliary planes cutting the surfaces of cylinders along the generatrices will be parallel to the horizontal trace  $M_1M'_1$  of plane  $QMM'$ .

The trace  $k_1$  of the first extreme auxiliary plane is drawn as a tangent to the base of the first cylinder. The trace  $q_1$  of the other extreme auxiliary plane is drawn as a tangent to the base of the second cylinder. With the help of these two planes we obtain the pairs of points  $1, 2$  and  $20, 21$  on the line of intersection of the cylinders  $I$  and  $II$ . Other points on this line are found with the help of a series of planes situated between the first and the last planes. Each of these planes gives us four points on the required line of intersection of the cylinders.

Having the horizontal projections of points  $1, 2, 3, \dots, 21$  we find their frontal projections by means of projectors and the

generatrices of the first cylinder, as shown in Fig. 361 by arrows for point  $1_2$ .

Both the sequence of joining up the corresponding projections of the points of the curve of intersection and the visibility of separate points, when projected onto surfaces  $\Pi_1$  and  $\Pi_2$ , are determined by the method of similar denotations. For this purpose

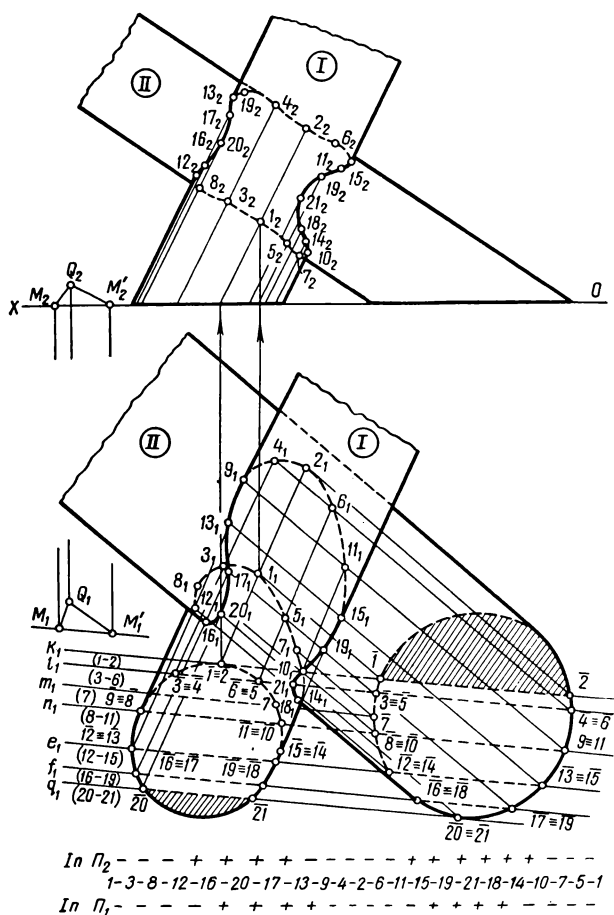


Fig. 361

points  $1, 2, 3, \dots, 21$  are projected obliquely onto the bases of the cylinders parallel to their generatrices. The notations of the points are written out in one row,  $1-3-8-12-16-20-17-13-9-4-2-6-11-15-19-21-18-14-10-7-5-1$ , since the line of intersection of the cylinders is a single closed curve.

6. A scheme of the possible positions of the extreme traces of simple auxiliary planes relative to the bases of the intersecting bodies is shown in Fig. 362, *a*.

Trace  $k_1$  of one extreme plane is a tangent to the base of the first body and intersects the base of the second body in two points. The trace  $q_1$  of the other extreme plane intersects the base of the first

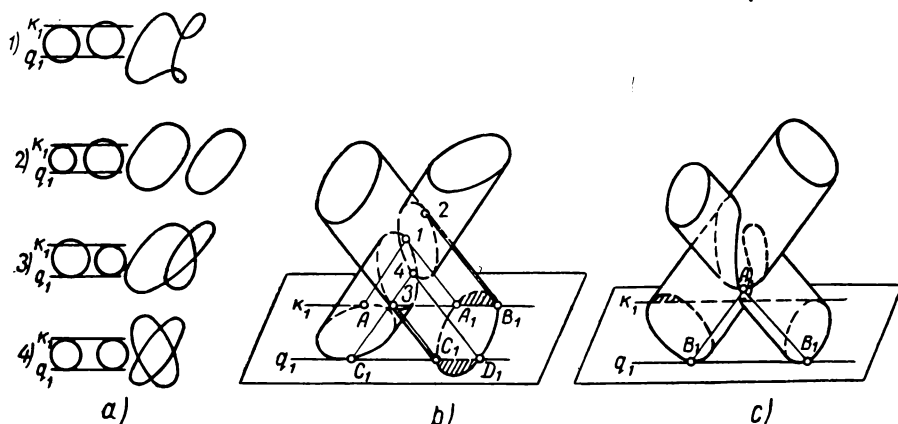


Fig. 362

body in two points and is a tangent to the base of the second. This is case 1, when the two bodies intersect each other in a single closed curve. The intersection of the cylinders, shown in Fig. 361, is an example of such a case.

If the traces of the two extreme auxiliary planes are tangents to the base of one body and cut the base of the other, the two surfaces intersect in two closed curves. This is case 2, an example of which is seen in Fig. 362, *b*.

When the trace of one extreme plane touches the bases of both bodies and the trace of the other extreme plane touches the base of the first body, but cuts the base of the second, the two curves of intersection of the surfaces have one common point. An example of this case 3 is seen in Fig. 362, *c*.

Finally, if the traces of the extreme planes touch the bases of both bodies, the surfaces intersect in two plane curves having two common points. This is case 4, of which the intersection of the surfaces of cylinders, shown in Fig. 353, *c*, may serve as an example.

7. In the constructions considered above, when simple oblique auxiliary planes were used to obtain points in the lines of intersection of surfaces, the bases of the latter were taken as lying in one plane. The construction of the line of intersection of two elliptical

cones, the bases of which are situated in different planes, is shown in Fig. 363. The base of cone *I* is contained in the horizontal plane of projection  $\Pi_1$  and the base of cone *II* in a plane parallel to  $\Pi_1$ . The auxiliary cutting planes are drawn through line *ST* joining

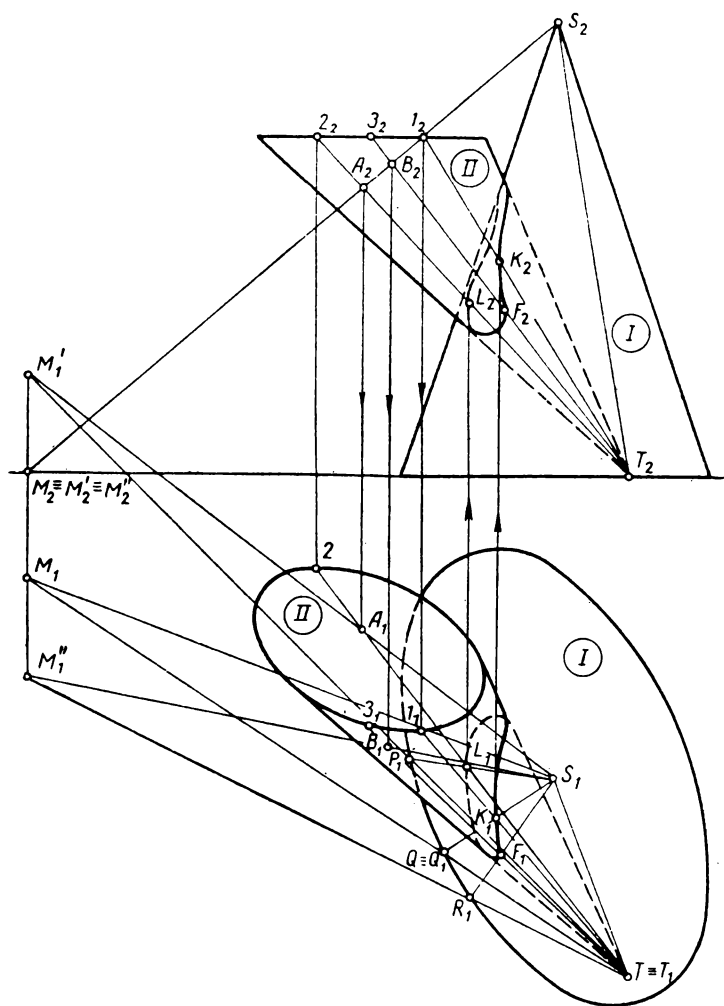


Fig. 363

the cone vertices, therefore these planes intersect both conical surfaces along their generatrices.

Let us examine the construction of a point *K* belonging to the line of intersection of the cones.

Line  $ST$  and line  $SI$ , which cuts  $ST$  and passes through a point  $I$  on the base of the second cone, determine the simplest auxiliary plane  $\alpha$ . The horizontal trace of plane  $\alpha$  passes through the horizontal piercing point  $M_1$  of line  $SI$  and through point  $T \equiv T_1$  which is the horizontal projection of vertex of cone  $II$ . Point  $Q \equiv Q_1$ , at the intersection of the horizontal projection of the base of cone  $I$  with the trace  $M_1T_1$ , is joined to point  $S_1$ . Point  $I_1$  is joined to point  $T_1$ . These two straight lines are the horizontal projections of the generatrices  $SQ$  and  $TI$  contained in plane  $\alpha$ . The point  $K_1$ , at the intersection of projections  $S_1Q_1$  and  $T_1I_1$  of these generatrices, is the horizontal projection of point  $K$ . Having point  $K_1$ , with the help of a projector we find point  $K_2$  on  $T_2I_2$ .

Points  $L$  and  $F$  are found by a similar construction using auxiliary planes  $\alpha'$  and  $\alpha''$  passing through line  $ST$  and through points  $A$  and  $B$  located on the generatrices  $T_2$  and  $T_3$  of cone  $II$  respectively. To simplify the drawing, when determining the projections  $A_2$  and  $B_2$ , the points  $A$  and  $B$  are taken so that the lines  $S_2A_2$  and  $S_2B_2$  coincide with line  $S_2I_2$ .

Curves are drawn through the corresponding projections of points  $K$ ,  $L$ ,  $F$  and of other points not shown in the drawing. These curves are the lines of intersection of the two given elliptic cones.

#### § 41. The Method of Spherical Sections

1. If a surface of revolution, the axis of which is parallel to the plane of projection, is cut by a sphere with its centre on the axis of revolution, the circle formed by the intersection of these two surfaces will be projected onto the given plane of projection as a straight line perpendicular to the corresponding projection of the axis of revolution.

Such a case is shown in Fig. 364, where a sphere ( $I$ ) with its centre at  $C$  intersects a cone ( $II$ ), a cylinder ( $III$ ) and a surface of revolution ( $IV$ ), the axes of which are parallel to the corresponding planes of projection and pass through the centre of the sphere  $C$ . The axes of the cone and cylinder are parallel to planes  $\Pi_1$  and  $\Pi_3$ . It follows that the circles, obtained by the intersection of the cylinder and the cone with the sphere, are projected onto planes  $\Pi_1$  and  $\Pi_3$  as straight lines perpendicular to the projections of the axes. The axis of the body of revolution  $IV$  is parallel to planes  $\Pi_1$  and  $\Pi_2$ . It follows that the circle of intersection of this surface with the sphere is projected onto  $\Pi_1$  and  $\Pi_2$  as a straight line, in each case perpendicular to the corresponding projections of the axis of revolution.

The relationships outlined above can be used for the construction of the line of intersection of a cylinder and a cone, the axes of which are parallel to plane  $\Pi_2$  and intersect in point  $C$ , as shown in Fig. 365. Let us assume point  $C$  as the centre of a sphere with



radius  $r$  and construct the frontal projections of auxiliary circles in which the cylinder and cone are intersected by the spherical surface. Both circles are projected onto plane  $\Pi_2$  as straight lines  $A_2B_2$  and  $D_2E_2$ , respectively perpendicular to the frontal projections of the axes of the cylinder and cone.  $A_2, B_2, D_2$  and  $E_2$  are the points of intersection of the frontal projections of the limiting elements of the cylinder and cone with the circle of radius  $r$  in which the sphere is projected onto plane  $\Pi_2$ . Since the auxiliary circles obtained lie on the surface of one and the same sphere, point  $2_2$  in which lines  $A_2B_2$  and  $E_2D_2$  intersect, will determine the frontal projections of the two points 2 lying on the line of intersection of the cylinder and cone.

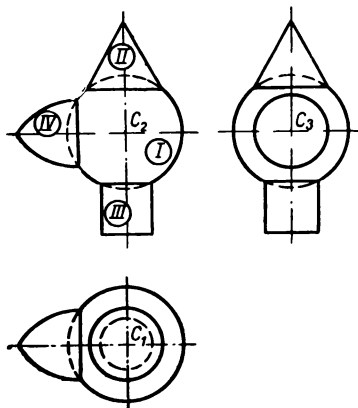


Fig. 364

Other auxiliary spheres can be described having the same centre  $C$  in order to find a series of points on the lines of intersection of the surfaces of the cylinder and cone.

The frontal projections  $1_2$  and  $4_2$  of points 1 and 4, in which the upper generatrix of the cylinder intersects the upper and lower generatrices of the cone, also lie on the frontal projection of line of transition, since these generatrices are situated in the same frontal plane passing through the axes of the cylinder and the cone. A curve is drawn through points  $1_2, 2_2, 3_2$  and  $4_2$ . This is the frontal projection of the line of intersection of the cylinder

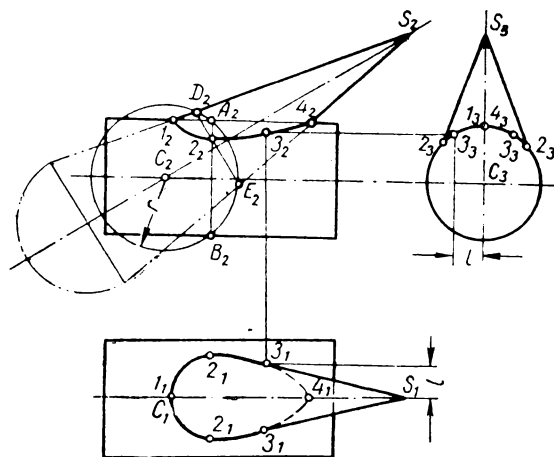


Fig. 365

and the cone. The profile projection  $2_3-3_3-1_3-4_3-3_3-2_3$  of the line of intersection coincides with the circle at which the surface of the cylinder is projected onto plane  $\Pi_3$ .

The horizontal projection of the line of intersection is constructed using the frontal and profile projections already obtained. Once points  $1_2$  and  $4_2$  are known points  $1_1$  and  $4_1$  can be found with the help of projectors.

It should be noted that the radius of the largest auxiliary sphere should be slightly less than the distance from centre  $C_2$  to the frontal projection of the point on the transition line farthest away from  $C_2$ , in this case,  $4_2$ . The radius of the smallest sphere should be equal to the distance from centre  $C_2$  to the frontal projection of the most distant generatrix.

2. Let it be required to determine the intersection of a cone and an annular torus (Fig. 366).

The basic points of the intersection of the given surfaces lie at the intersection of the limiting elements of the cone with the outside generatrix of the torus. The highest point is  $1_2$ , the lowest being  $2_2$ .

The frontal plane passing through the axis of the cone is at the same time the plane of symmetry of the torus, so it is convenient to use spherical sections to determine intermediate points belonging to the line of intersection of the cone and torus.

Let us subdivide that portion of the torus limited by planes  $\alpha$  and  $\alpha'''$  into separate parts with the help of frontal-projecting planes  $\alpha, \alpha', \alpha'', \alpha'''$  passing through the centre  $O$  of the torus. These parts may be called "curved cylinders". We can assume the surface of each part to be the surface of a right circular cylinder touching a curved cylinder in the circle of intersection of the torus by a plane. We are then able to construct the points of intersection of these straight cylinders with the cone.

Let us follow through the construction of a point  $3_2$  belonging to the intersection of the cone and the cylinder lying between planes  $\alpha$  and  $\alpha'$ . The axis of this cylinder intersects the axis of the cone in point  $C_2$ . With point  $C_2$  as the centre we describe a circle, the radius of which is equal to the distance from  $C_2$  to point  $K_2$  in which trace  $\alpha'_2$  intersects the outside generatrix of the torus. This

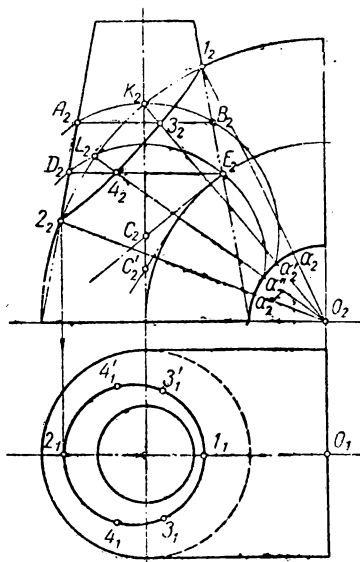


Fig. 366

circle is the frontal projection of the auxiliary spherical surface intersecting the cone and the assumed right cylinder contained between planes  $\alpha'$  and  $\alpha''$ . The frontal projection of the circular section of the cone is line  $A_2B_2$  and it is perpendicular to the frontal projection of the axis of the cone. The frontal projection of the circle of intersection of the cylinder coincides with trace  $\alpha'_2$ . The intersection of these lines gives point  $3_2$  contained in the frontal projection of the line of intersection of the cone and the torus.

Point  $4_2$  is found in a similar manner by drawing a circle with radius  $C'_2L_2$  having  $C'_2$  as its centre. Point  $4_2$  lies at the intersection of lines  $D_2E_2$  and  $L_2O_2$ .

The construction of the horizontal projections  $3_1$ ,  $3'_1$ ,  $4_1$  and  $4'_1$  of points  $3$ ,  $3'$ ,  $4$  and  $4'$  is carried out by drawing the circular sections of the cone given by the horizontal planes, the frontal traces of which pass through points  $3_2$  and  $4_2$  respectively. From points  $3_2$  and  $4_2$  projectors are drawn to intersect the horizontal projections of these circles. Having obtained points  $3_1$ ,  $3'_1$ ,  $4_1$  and  $4'_1$ , we draw a curve through points  $1_1$ ,  $3_1$ ,  $4_1$ ,  $2_1$ ,  $4'_1$  and  $3'_1$ . This curve is the horizontal projection of the intersection of the cone and torus.

## § 42. Transformation of Projections and Oblique Projection Used to Determine the Intersection of Surfaces

1. The transformation of projections for constructing the intersection of two surfaces, say a prism and sphere (Fig. 367), consists in obtaining a projection of the two surfaces in which the line of intersection coincides with the projection of one of the given surfaces, in this case, of the prism. This may be accomplished by successive revolutions of the given bodies about the corresponding axes, or by successive substitutions of the planes of projection.

In the multiview drawing (Fig. 367) the problem is solved by substituting the planes of projection. Plane  $\Pi_2$  is substituted by plane  $\Pi_4$  perpendicular to plane  $\Pi_1$  and parallel to the lateral edges of the prism. These edges will be projected onto plane  $\Pi_4$  in their true length. The second new plane  $\Pi_5$ , perpendicular to plane  $\Pi_4$ , is also perpendicular to the lateral edges of the prism. The prism is then projected onto this plane  $\Pi_4$  as the triangle  $A_5B_5D_5$ . The sides of this triangle represent the projection of the faces of the prism. The part  $A_5\delta_5$  of side  $A_5B_5$  of the triangle is the projection on plane  $\Pi_5$  of the arc of the circle in which face  $AB$  of the prism cuts the surface of the sphere.

Similarly,  $A_5\delta_5$  is the projection on plane  $\Pi_5$  of the arc of the circle in which face  $AD$  of the prism cuts the sphere. Since side  $B_5D_5$  of the triangle does not intersect the circle having  $C_5$  as its centre, it follows that face  $BD$  of the prism does not cut the surface of the sphere.

Let us now take a series of points on the sides  $A_5B_5$  and  $A_5D_5$

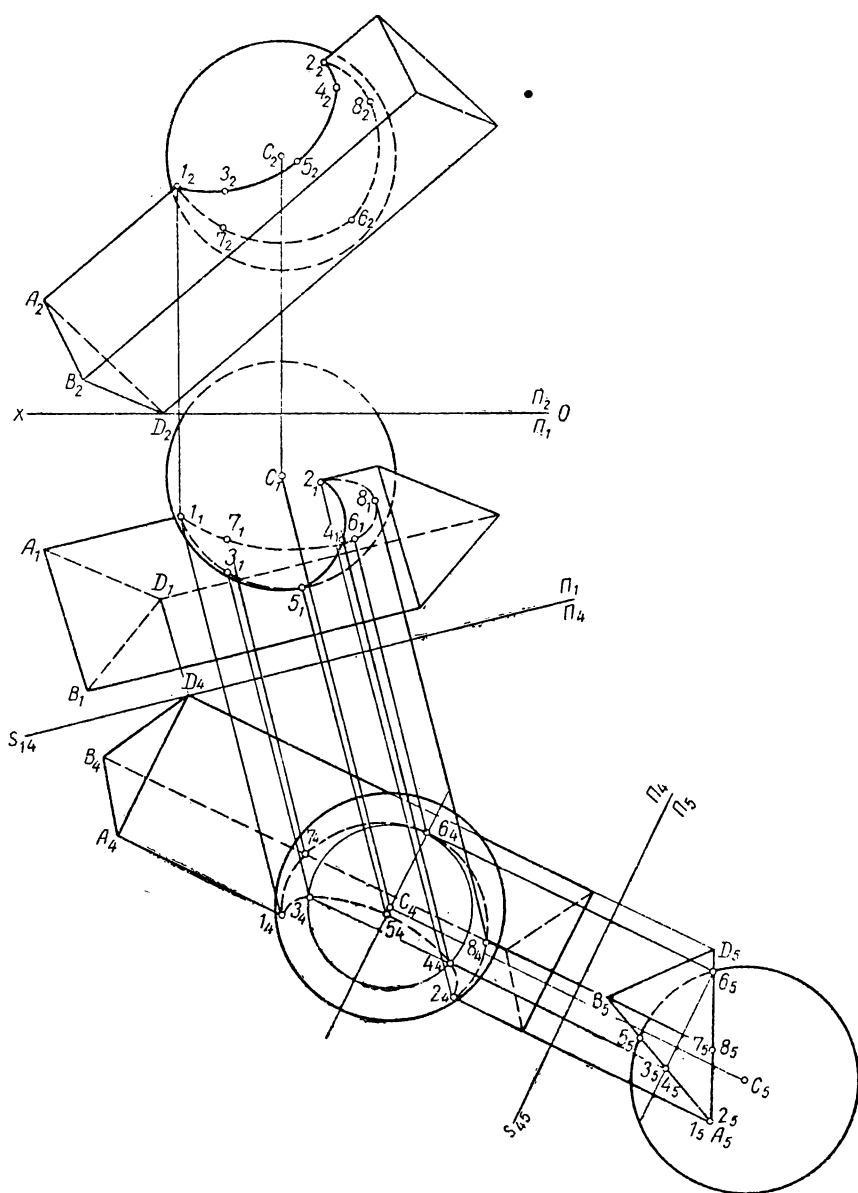


Fig. 367

of the triangle and, using the method of inverse projection, find the projections of these points, first in the system of planes  $\Pi_1$ - $\Pi_4$  and then in the principal system of planes of projection  $\Pi_1$ - $\Pi_2$ . Points 5 and 6 are special points. The edge  $A$  of the prism intersects the sphere in points 1 and 2. Points 3, 4, 7 and 8 are intermediate points.

The projections of points 1, 2, 3, 4, 5, 6, 7 and 8 on plane  $\Pi_4$  are obtained by cutting the sphere with planes parallel to the plane of projection  $\Pi_4$ . This construction is shown for points 3<sub>4</sub>, 4<sub>4</sub> and 6<sub>4</sub>. The projections of the points on planes  $\Pi_1$  and  $\Pi_2$  are found by means of their projections on planes  $\Pi_4$  and  $\Pi_5$ . It should be remembered that the distances of these points from the respective planes of projection remain unchanged. The two arcs of circles in which the sphere is cut by the prism faces  $AB$  and  $AD$  are projected onto planes  $\Pi_4$ ,  $\Pi_1$  and  $\Pi_2$  as arcs of ellipses.

2. The construction of the intersection of two oblique prisms using oblique projection is shown in the multiview drawing (Fig. 368). The prisms are projected obliquely onto plane  $\Pi_1$  in direction  $s$  parallel to the lateral edges  $DK$ ,  $EL$  and  $FM$  of the second prism.

The oblique projection of prism  $II$  on plane  $\Pi_1$  coincides with the orthographic projection  $K_1L_1M_1$  of its base  $KLM$  on that same plane  $\Pi_1$ . The oblique projection of prism  $I$  on plane  $\Pi_1$  is shown in the drawing by dots and dashes. The oblique projections of its vertices are found in the same way as the horizontal traces of oblique lines are found. This construction is shown for vertex  $A$ .

The intersection of the oblique projections of the lateral edges of the first prism with the sides of triangle  $K_1L_1M_1$ , i. e., with the oblique projection of prism  $II$ , determines points  $\bar{1}_1, \bar{2}_1, \bar{3}_1, \bar{4}_1, \bar{5}_1, \bar{6}_1, \bar{7}_1$  and  $\bar{8}_1$ . These points are the oblique projections of points 1, 2, 3, 4, 5, 6, 7 and 8, contained in the line of intersection of the prisms.

By inverse projection points  $1_1, 2_1, 3_1, 4_1, 5_1$  and  $6_1$  are found at the intersection of the horizontal projections of the oblique projectors with the orthographic projections on plane  $\Pi_1$  of the corresponding edges of the first prism. In order to determine points  $7_1$  and  $8_1$  two auxiliary lines  $S7$  and  $P8$  are used. These lines are first drawn on the oblique projection of the prism through the duplicate point  $\bar{7}_1 \equiv \bar{8}_1$ . Then they are drawn on the projection of prism  $II$  on plane  $\Pi_1$ . This construction is indicated by arrows. The frontal projections  $1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2$  and  $8_2$  of all the eight points found are constructed by passing projectors through points  $1_1, 2_1$ , etc., to intersect the frontal projections of the respective lateral edges of the prism.

3. The line of intersection of a cylinder and a pyramid (Fig. 369) may also be constructed by the method of oblique projection of both

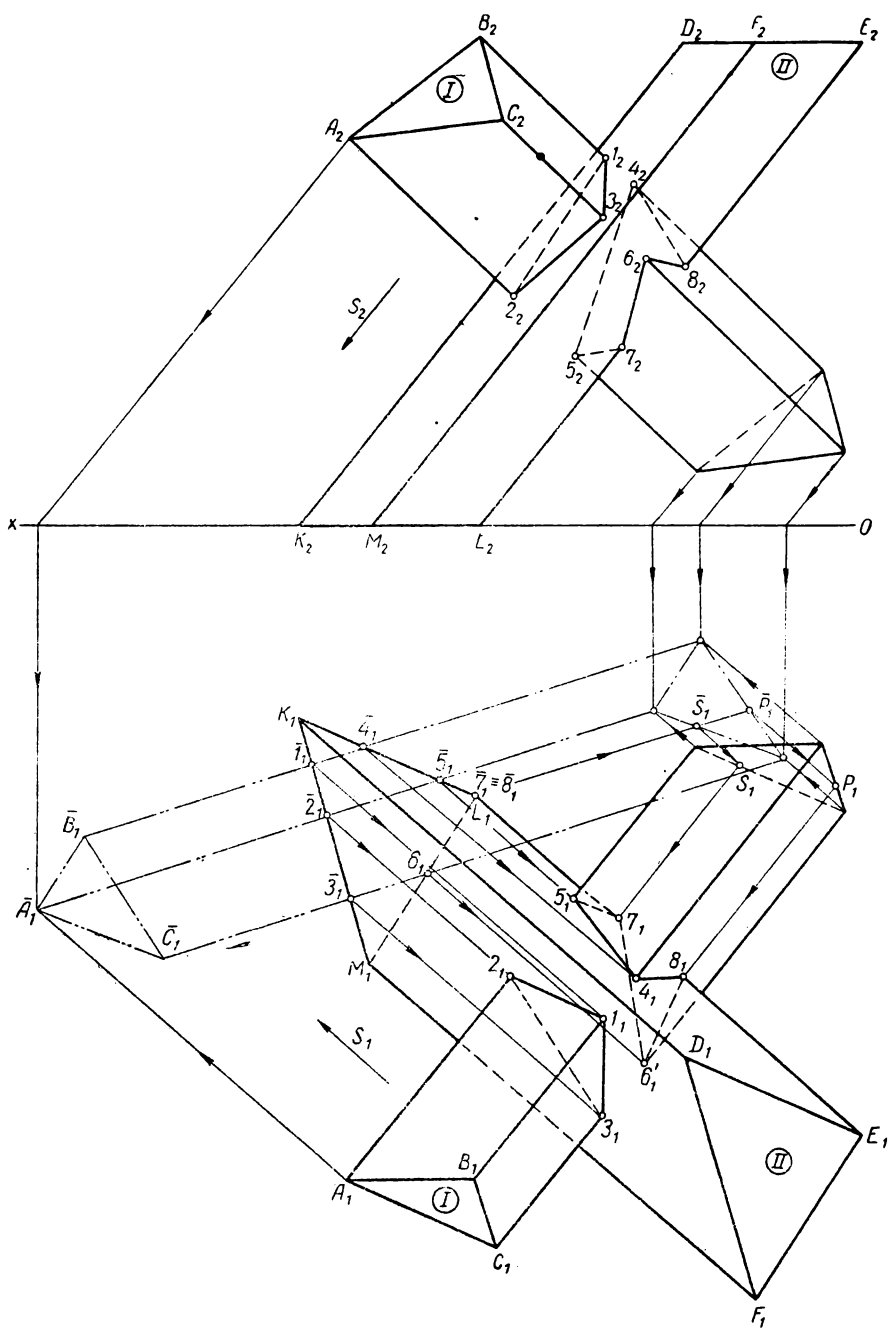


Fig. 368

bodies onto plane  $\Pi_1$  in direction  $l$  parallel to the generatrices of the cylinder. The oblique projection of the cylindrical surface coincides with the orthographic projection on plane  $\Pi_1$  of the base of the cylinder itself. The oblique projection of the base of the pyramid coincides with the orthographic projection of its base on plane  $\Pi_1$ . Next point  $\bar{S}_1$ , the oblique projection on plane  $\Pi_1$  of the

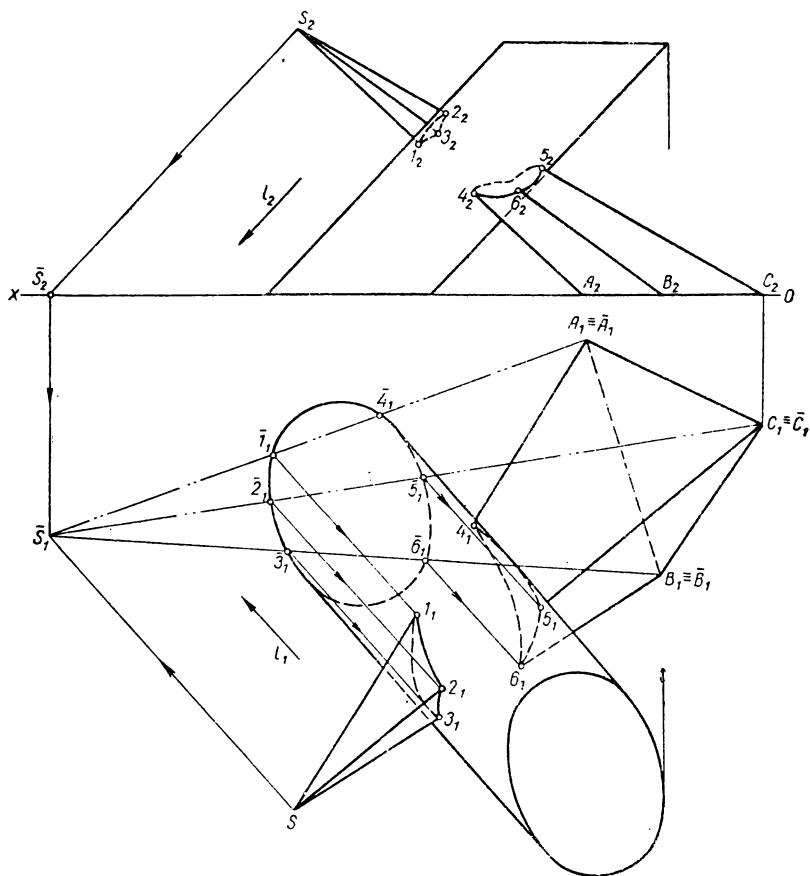


Fig. 369

vertex of the pyramid  $S$ , is found. Point  $\bar{S}_1$  is joined by straight lines with points  $A_1 \equiv \bar{A}_1$ ,  $B_1 \equiv \bar{B}_1$  and  $C_1 \equiv \bar{C}_1$ . This gives us the oblique projection  $\bar{S}_1 \bar{A}_1 \bar{B}_1 \bar{C}_1$  of the pyramid on plane  $\Pi_1$ . The intersection of the projections  $\bar{S}_1 \bar{A}_1$ ,  $\bar{S}_1 \bar{B}_1$  and  $\bar{S}_1 \bar{C}_1$  of the edges of the pyramid with the ellipse, which is the oblique projection of the cylinder and coincides with the orthographic projection of the base

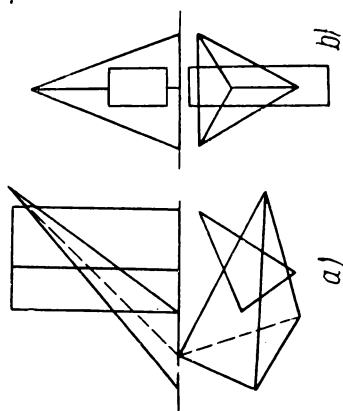


Fig. 370

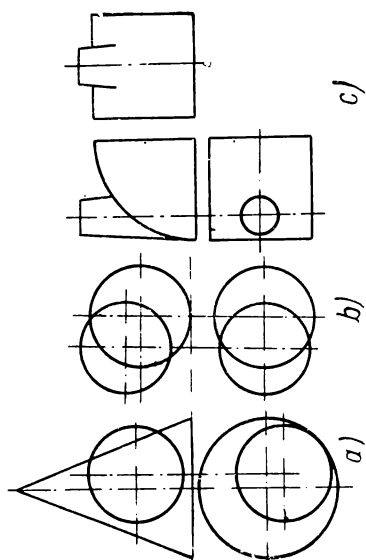


Fig. 372

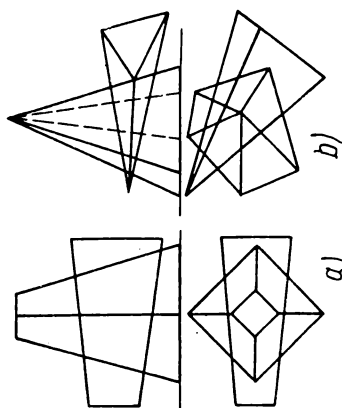


Fig. 371

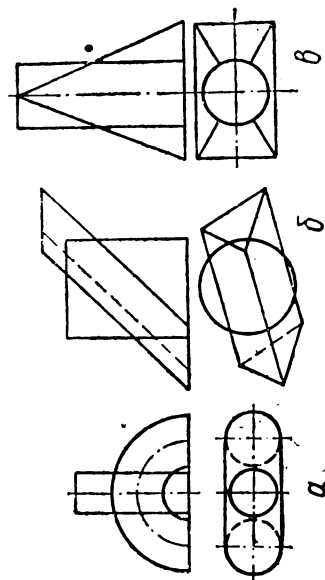


Fig. 373



of the cylinder on  $\Pi_1$ , determines the oblique projections  $\overline{1}_1$ ,  $\overline{2}_1$ ,  $\overline{3}_1$ ,  $\overline{4}_1$ ,  $\overline{5}_1$  and  $\overline{6}_1$  of the points in which the edges of the pyramid intersect the cylinder. The inverse projecting of these points contained in the intersection of the cylinder and pyramid is carried out the same way as in the previous example (see Fig. 368).

### Problems and Exercises

1. How is the line of transition, or intersection of surfaces of two polyhedra constructed?

2. Describe how the method of similar notations is used to determine the intersection of two surfaces?

3. What types of auxiliary planes are used when constructing points contained in the line of intersection of two surfaces?

4. What is the procedure for constructing the intersection of two surfaces by transforming their projections?

What is the procedure when using oblique projection for this purpose?

5. Construct the intersection of the pyramid and the prism, shown in Fig. 370.

6. Construct the intersection of the pyramids, shown in Fig. 371.

*Hint.* In the first example (Fig. 371, *a*) the section planes used should be projecting planes. In the second case (Fig. 371, *b*) simple oblique section planes passing through the vertices of the pyramid should be used (Fig. 360, *c*).

7. Construct the intersection of the following surfaces: a) a cone and sphere, b) two spheres, c) a cone and a cylinder (Fig. 372).

8. Construct the intersection of the following surfaces: a) a torus and a cylinder, b) a cylinder and a prism, c) a cylinder and a pyramid (Fig. 373).

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## Chapter XI

### AXONOMETRIC PROJECTION

#### § 43. General

The orthographic projections of different objects completely determine their form, their dimensions, and their relative position in space. Such projections are convenient to use since two dimensions, parallel to the respective axes of projection, the ground lines, are shown without distortion while the third dimension is represented as a point. However, orthographic projections are not completely descriptive since they depict the objects not as one whole, but by means of their separate projections. From these drawings it is necessary to imagine the reconstruction of the form of the objects in space. That is why not only orthographic projections, but also axonometric projections, are widely used.

The main advantage of axonometric projections consists in their clarity. Because of this, and also due to the inherent advantage of axonometric projections, i. e., the ease with which measurements are made on the projections, they are gaining popularity in engineering. In Fig. 374 a detail, the body of a valve, is depicted both in orthographic (Fig. 374, *a*) and in axonometric projection (Fig. 374, *b*). The axonometric representation of any detail is obviously more descriptive.

When constructing axonometric projections the objects are considered as being situated in a system of rectangular coordinate axes. The objects, together with the axes, are projected onto a selected plane of projection. The latter is called the *plane of axonometric projection*. The projections obtained on this plane are called *axonometric projections*.

The projections of the coordinate axes on the plane of axonometric projection are called *axonometric axes*.

Fig. 375 shows a plane of axonometric projection  $\Pi'$  arbitrarily located in respect to the trihedral formed by the coordinate planes  $Oxz$ ,  $Oxy$  and  $Oyz$ . The axonometric axes  $O'x'$ ,  $O'y'$  and  $O'z'$  obtained in plane  $\Pi'$  by projecting the rectangular coordinate axes

in the direction  $s$ , and the axonometric projection  $A'$  of point  $A$ , are also shown.

The projector  $s$  may be either perpendicular or inclined to plane  $\Pi'$ . In the first case the axonometric projections are called *rectangular* and in the latter — *oblique*. The former are also referred to as right or right-angle projections.

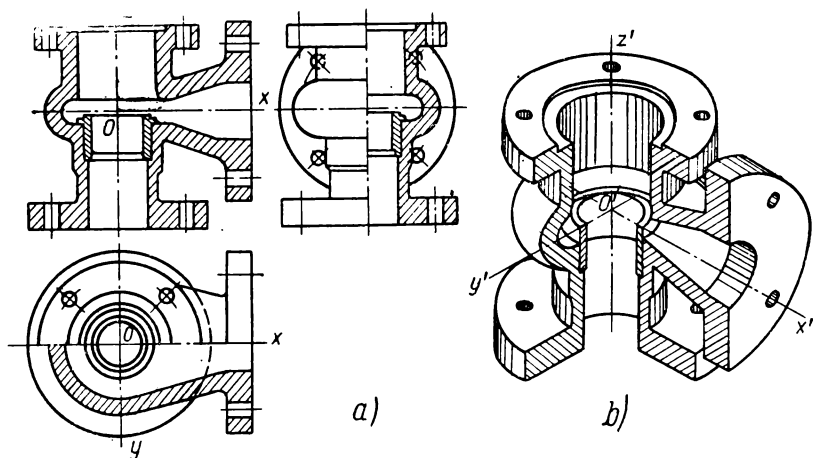


Fig. 374

The rectangular and oblique axonometric projections also differ in the degree of distortion of dimensions. The *factor*, or *index*, of *distortion* or *foreshortening* is the ratio of the length of the axonometric projection of a line to the length of the line itself.

Let the coordinate angle  $Oxyz$  (Fig. 376) be projected onto the plane of axonometric projection  $\Pi'$  in the direction  $s$ , and let point  $O'$  be the axonometric projection of the point of origin  $O$ . Then line  $O'x'$  will be the axonometric projection of  $Ox$ . The lines  $O'y'$  and

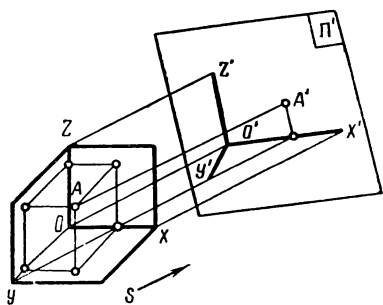


Fig. 375

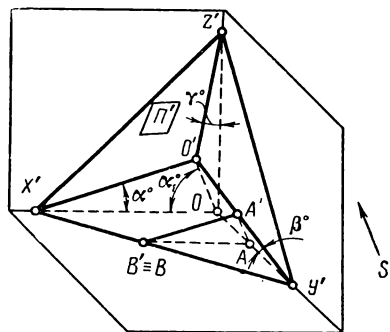


Fig. 376

$O'z'$  will be the axonometric projections of  $Oy$  and  $Oz$ . The ratios  $\frac{O'x'}{Ox} = p$ ,  $\frac{O'y'}{Oy} = q$  and  $\frac{O'z'}{Oz} = r$  are the factors of distortion determining the degree of foreshortening in the directions of the axes  $Ox$ ,  $Oy$  and  $Oz$ .

Let us take a line  $AB$  in the plane  $Oxy$ , parallel to axis  $Ox$ . Since parallel projection is used, the axonometric projection  $A'B'$  is parallel to the axonometric axis  $O'x'$ . It follows that  $\frac{A'B'}{AB} = \frac{O'x'}{Ox} = p$ .

When all three factors of distortion are equal, i. e.,  $p=q=r$ , the axonometric projection is called *isometric* projection and the scale used is an isometric scale.

Axonometric projection with two equal factors of distortion and the third differing is called *dimetric* projection, i. e., two measurements are equal. The scale used is a dimetric scale. If there are no equal factors axonometric projection is called *trimetric* projection, i. e., possessing three different measurements. The scale then used is called a trimetric scale.

The direction of the projection for the construction of axonometric projections is of considerable importance. This direction should be chosen so that one axonometric projection gives a complete idea of the form of the object represented. The direction of projection should not be perpendicular to any of the coordinate planes, otherwise one of the coordinate axes will be projected onto plane  $\Pi'$  as a point and the axonometric representation will lack clarity.

The plane of axonometric projection  $\Pi'$  should also be given a suitable position in space with respect to the coordinate planes  $Oxz$ ,  $Oyx$  and  $Oyz$ . In rectangular axonometric projection plane  $\Pi'$  must cut all three coordinate axes, otherwise the projectors perpendicular to  $\Pi'$  may cause one of the axes  $x$ ,  $y$  or  $z$  to be projected onto  $\Pi'$  as a point. They may also project two of the axes as a single line. For example, if plane  $\Pi'$  is perpendicular to plane  $Oxz$  and to axis  $Oz$  the axis  $Ox$  will be projected onto  $\Pi'$  as a point and the axonometric representation will not be sufficiently clear.

In oblique axonometric projection, when the position of the plane of projection does not depend on the choice of direction of the projection and, conversely, the selection of the direction of projection does not depend on the position of the plane of axonometric projection, plane  $\Pi'$  may be parallel to one of the coordinate axes or coordinate planes, or it may not intersect all three coordinate axes. In oblique axono-

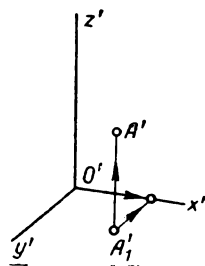


Fig. 377

metric plane  $\Pi'$  is usually assumed as parallel to one of the coordinate planes.

The relative positions of the plane of axonometric projection and the direction of projection in space determine the factors of distortion and the angles between the axonometric axes.

The construction of a dimetric rectangular projection  $A'$  of point  $A$  with respect to the axes  $O'x'$ ,  $O'y'$  and  $O'z'$  inclined at angles  $7^\circ$ ,  $41^\circ$  and  $90^\circ$  to the horizontal lines is shown in Fig. 377. The sequence of construction is shown by arrows. First, from point  $O'$  we lay off the value  $x$  along axis  $O'x'$ . Then parallel to the axis  $O'y'$  we lay off the value  $y$ . This gives us point  $A_1^*$ . Finally, along the vertical, upwards from point  $A_1^*$ , the value  $z$  is laid off. This gives us the rectangular dimetric projection  $A'$  of point  $A$ .

#### § 44. The Elements of Rectangular Axonometric Projection

1. The determination of the factors of distortion. If the projections are perpendicular to the plane of axonometric projection  $\Pi'$  (Fig. 376) and point  $O'$  is the axonometric projection of point  $O$ , then line  $O'O$  is perpendicular to plane  $\Pi'$  and the lines  $O'x'$ ,  $O'y'$  and  $O'z'$  will be the sides of the right triangles  $O'x'O$ ,  $O'y'O$  and  $O'z'O$ . Therefore:  $\frac{O'x'}{Ox} = \cos \alpha^\circ$ ;  $\frac{O'y'}{Oy} = \cos \beta^\circ$ ;  $\frac{O'z'}{Oz} = \cos \gamma^\circ$ .

The left parts of the equations represent the factors of distortion:

$$\frac{O'x'}{Ox} = p; \quad \frac{O'y'}{Oy} = q; \quad \frac{O'z'}{Oz} = r.$$

Therefore:  $\cos \alpha^\circ = p$ ;  $\cos \beta^\circ = q$ ;  $\cos \gamma^\circ = r$ .

Let the coordinates of point  $O'$  be  $x$ ,  $y$ ,  $z$  (Fig. 378). Let the length of  $O'O$  be  $k$ , then

$$x^2 + y^2 + z^2 = k^2. \quad (1)$$

But  $x = k \cos \delta^\circ$ ;  $y = k \cos \lambda^\circ$ ;  $z = k \cos \varphi^\circ$ .

Substituting these values of the coordinates of point  $O'$  in equation (1), we obtain

$$k^2 \cos^2 \delta^\circ + k^2 \cos^2 \lambda^\circ + k^2 \cos^2 \varphi^\circ = k^2 \text{ and } \cos^2 \delta^\circ + \cos^2 \lambda^\circ + \cos^2 \varphi^\circ = 1. \quad (2)$$

So the sum of the squares of the cosines of the angles made with the coordinate axes by the line  $O'O$  passing through the origin of the coordinates is equal to unity.

The line  $O'O$  (Fig. 376) forms angles  $\alpha_1^\circ$ ,  $\beta_1^\circ$  and  $\gamma_1^\circ$  with axes  $Ox$ ,  $Oy$  and  $Oz$ .

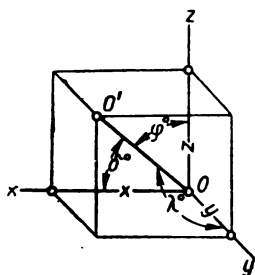


Fig. 378

\* This point  $A_1^*$  is called the secondary projection.

These angles are complementary to angles  $\alpha^\circ$ ,  $\beta^\circ$ ,  $\gamma^\circ$ . In Fig. 376 the only notation of these angles shown is that of the angle  $\alpha_1^\circ$  between the ground line  $Ox$  and the line  $O'O$ . From what has been said it follows:

$$\cos^2 \alpha_1^\circ + \cos^2 \beta_1^\circ + \cos^2 \gamma_1^\circ = 1 \quad (3)$$

since

$$\alpha^\circ = (90^\circ - \alpha_1^\circ), \beta^\circ = (90^\circ - \beta_1^\circ), \gamma^\circ = (90^\circ - \gamma_1^\circ),$$

we have

$$\cos \alpha_1^\circ = \sin \alpha^\circ; \cos \beta_1^\circ = \sin \beta^\circ; \cos \gamma_1^\circ = \sin \gamma^\circ.$$

Hence,

$$\sin^2 \alpha^\circ + \sin^2 \beta^\circ + \sin^2 \gamma^\circ = 1. \quad (4)$$

But

$$\sin^2 \alpha^\circ = 1 - \cos^2 \alpha^\circ; \sin^2 \beta^\circ = 1 - \cos^2 \beta^\circ; \sin^2 \gamma^\circ = 1 - \cos^2 \gamma^\circ. \quad (5)$$

Substituting (5) in (4) we obtain

$$\cos^2 \alpha^\circ + \cos^2 \beta^\circ + \cos^2 \gamma^\circ = 2.$$

It has already been demonstrated that  $\cos \alpha^\circ = p$ ;  $\cos \beta^\circ = q$ ;  $\cos \gamma^\circ = r$ .

Therefore:  $p^2 + q^2 + r^2 = 2$ , i. e., the sum of the squares of the factors of distortion in a rectangular axonometric system is equal to two.

In a right isometric system the distortion factors along the three axes are equal to each other:  $p = q = r$ . Therefore,  $3p^2 = 3q^2 = 3r^2 = 2$ .

It follows:  $p = q = r = \sqrt{\frac{2}{3}} = 0.8165 \approx 0.82$ .

Because of this, in a right isometric system, the lengths of projections of parts of lines, i. e., of segments, located along axes  $x$ ,  $y$  and  $z$  or parallel to them, are equal to 0.82 of the real length of the lines which have been projected.

In a right dimetrical system of projections the distortion factors  $p$  and  $r$  along axes  $x$  and  $z$  are taken as equal while the factor  $q$  along axis  $y$  is assumed to be equal to  $\frac{p}{2} = \frac{r}{2}$ .

Therefore, we have:

$$p^2 + \frac{1}{4} p^2 + p^2 = 2, \text{ or } p^2 = \frac{8}{9};$$

hence,

$$p = r = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} = 0.9428 \approx 0.94$$

and

$$q = \frac{p}{2} = \frac{0.9428}{2} = 0.4714 \approx 0.47.$$

2. The determination of the angles between axonometric axes. The triangle  $x'y'z'$  given by the intersection of the plane of axonometric projection  $\Pi'$  with the coordinate planes is called the *trace triangle* (Fig. 376). It is easy to determine the angles between the axonometric axes with the help of the trace triangle.

In right isometric projection  $p=q=r$ . Therefore,  $\cos \alpha^\circ = \cos \beta^\circ = \cos \gamma^\circ$  which means that when the angles are acute they themselves are equal, i. e.,  $\alpha^\circ = \beta^\circ = \gamma^\circ$ .

The line segment  $O'O$  is one of the sides of the right triangles  $O'x'O$ ,  $O'y'O$  and  $O'z'O$ . Therefore:

$O'O = Ox' \cdot \sin \alpha^\circ$ ;  $O'O = Oy' \cdot \sin \beta^\circ$ ;  $O'O = Oz' \cdot \sin \gamma^\circ$ . It follows that when angles  $\alpha^\circ$ ,  $\beta^\circ$  and  $\gamma^\circ$  are equal, the line segments  $Ox'$ ,  $Oy'$  and  $Oz'$  are also equal in length.

Since  $Ox'$ ,  $Oy'$  and  $Oz'$  are the sides of the other three right triangles  $xOz$ ,  $xOy$ ,  $zOy$ , the hypotenuses  $xy$ ,  $xz$  and  $zy$  of these triangles are also equal to each other, so the trace triangle is an equilateral triangle.

Angles  $x'O'z'$ ,  $x'O'y'$  and  $z'O'y'$  are also equal, since their sides  $O_1x'$ ,  $O_1y'$  and  $O_1z'$  coincide with the altitudes of the equilateral triangle. Each of these angles is  $120^\circ$ .

It follows that in isometric right projection the angles between the axes are  $120^\circ$ . When these axes are being constructed it is usual to make axis  $O'z'$  a vertical (Fig. 379).

In dimetric right projection the distortion factors  $p$  and  $r$  are equal and we have  $\cos \alpha^\circ = \cos \gamma^\circ$ . This means that the angles  $\alpha^\circ$  and  $\gamma^\circ$  are also equal (Fig. 376).

But  $O'O = Ox \cdot \sin \alpha^\circ$ ;  $O'O = Oz \cdot \sin \gamma^\circ$ .

Hence  $O'x' = O'z'$ . With  $O'x' = 1$  and  $O'z' = 1$  we obtain  $x'z' = \sqrt{2}$ .

Once the distortion factors for dimetric right projections have been determined it can be established that  $p=r = \frac{2\sqrt{2}}{3}$ . There-

fore,  $O'x' = O'z' = \frac{2\sqrt{2}}{3}$ .

Since segments  $O'x'$  and  $O'z'$  are equal, it follows that the sides

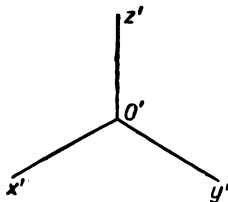


Fig. 379

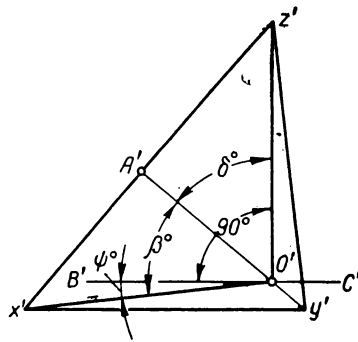


Fig. 380

$x'y'$  and  $y'z'$  of the trace triangle  $x'y'z'$  are also equal, i. e., this triangle is an isosceles triangle (Fig. 380).

The altitude  $O'A'$  divides the side  $x'z'$  of the triangle  $x'y'z'$  into two equal parts,  $A'x' = A'z'$ ; but  $x'z' = \frac{\sqrt{2}}{2}$ , therefore  $A'x' = \frac{\sqrt{2}}{2}$  and  $A'z' = \frac{\sqrt{2}}{2}$ .

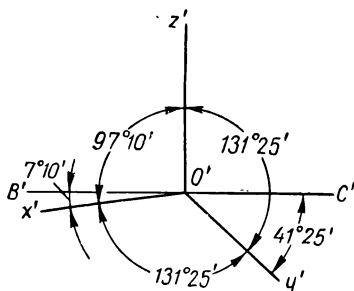


Fig. 381

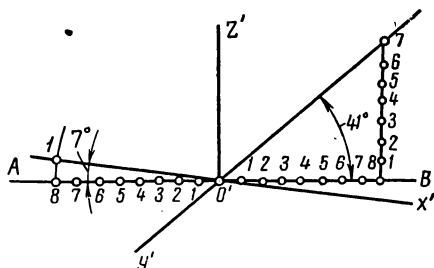


Fig. 382

Triangle  $A'z'O'$  is a right triangle, therefore we have:

$$\sin \delta^\circ = A'z' : O'z' = \frac{\sqrt{2}}{2} : \frac{2\sqrt{2}}{3} = 0.75.$$

The angle  $\delta^\circ$  whose  $\sin \delta^\circ = 0.75$  is  $48^\circ 35'$ . The angle  $x'O'z' = 2\delta^\circ = 48^\circ 35' \times 2 = 97^\circ 10'$ .

Angle  $A'O'B'$  is complementary to angle  $\delta$ . But angle  $A'O'B'$  is equal to angle  $C'O'y'$ . Since  $\delta^\circ = 48^\circ 35'$ , it follows that angle  $A'O'B' = \text{angle } C'O'y' = 90^\circ - 48^\circ 35' = 41^\circ 25'$ .

Summing up, the angle between axes  $O'z'$  and  $O'x'$  (Fig. 381) is  $97^\circ 10'$ , the angle between axes  $O'z'$  and  $O'y'$  is  $131^\circ 25'$ , and the angle between axes  $O'x'$  and  $O'y'$  is also  $131^\circ 25'$ . Since the axis  $O'z'$  is usually taken as vertical, it follows that in relation to the horizontal line  $B'C'$ , axis  $O'x'$  makes an angle of  $97^\circ 10' - 90^\circ = 7^\circ 10'$  (roughly  $7^\circ$ ) with  $B'C'$ , and axis  $O'y'$  makes an angle of  $131^\circ 25' - 90^\circ = 41^\circ 25'$  (roughly  $41^\circ$ ) with  $B'C'$ .

The construction of the dimetric axes is carried out as follows: First, the line  $AB$  is drawn (Fig. 382) and on it point  $O'$  is marked. This will be the axonometric projection of the coordinate axis. From this point axis  $O'z'$  is erected perpendicular to  $AB$ . Then on the straight line  $AB$ , to each side of point  $O'$ , eight equal segments of any suitable length are laid off. From the left point 8 a perpendicular to line  $AB$  is erected. On this line one of the segments is stepped off. From the right-hand point 8 a perpendicular to  $AB$  is erected and seven segments are stepped off on it. The points 1 and 7 obtained in this way are joined to point  $O'$  by straight lines. These are the axes  $O'x'$  and  $O'y'$  in the rectangular dimetric projection. This method is based on the



construction of an angle by its tangent ( $\tan 7^\circ 10' = \frac{1}{8}$ ;  $\tan 41^\circ 25' = \frac{7}{8}$ ).

Eight different ways in which the axonometric semi-axes can be seen in the various octants, depending on the position of the observer's eye, are shown in Fig. 383. If the eye of the observer is in the 1 or 4 or in the 6 or 7 octants, the axis  $O'x'$  will be directed to the left of point  $O'$  and the axis  $O'y'$  to the right of it. When the observer's eye, i. e., the point of sight, is in one of the rest octants,

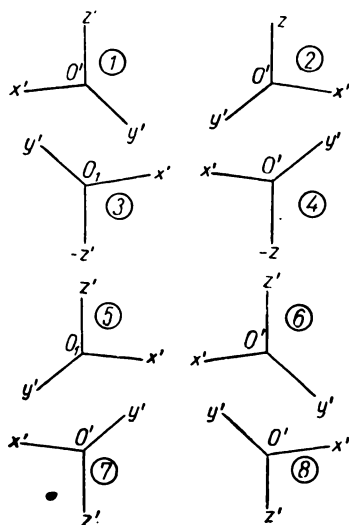


Fig. 383

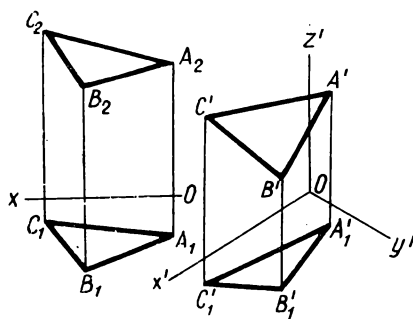


Fig. 384

2, 3, 5 and 8, the axis  $O'x'$  falls to the right of point  $O'$  and axis  $O'y'$  to the left.

In Figs. 379 and 381 the direction of the axes  $O'x'$  and  $O'y'$  corresponds to the case when the point of sight is in the 1 octant. In Fig. 382 the point of sight is in the 5 octant.

## § 45. The Construction of Rectangular Axonometric Projections

When constructing axonometric projections it is necessary to remember that the foreshortening or distortion dealt with in the previous paragraph concerns only those lines which lie on axes  $O'x'$ ,  $O'y'$  and  $O'z'$ , or lines which are parallel to these axes. The length of the axonometric projections of lines which are not parallel to the axes may be determined by constructing their extreme points by means of their coordinates.

The construction of the rectangular dimetric projection  $A'$  of point  $A$  by its given coordinates  $x, y, z$  is shown in Fig. 377. Using this technique it is always possible to construct the axonometric

projection of any object from its multiview drawing, assuming that the distortion of factors for the axonometric axes are given.

The representation in rectangular isometric projection of triangle  $ABC$ , given in the multiview drawing by projections  $A_1B_1C_1$  and  $A_2B_2C_2$ , is shown in Fig. 384. The construction is as follows: first, the axonometric coordinates  $x'$ ,  $y'$ ,  $z'$  of the vertices of the triangle are determined by multiplying the true coordinates  $x$ ,  $y$ ,  $z$  by 0.82. Then, using the coordinates  $x'$  and  $y'$  of the vertices  $A$ ,  $B$  and  $C$ , their secondary horizontal projections  $A'_1, B'_1$  and  $C'_1$  are constructed. Finally, from these points the values  $zA$ ,  $zB$ ,  $zC$  are laid off. The points  $A'$ ,  $B'$  and  $C'$  obtained are the axonometric projections of the vertices of the given triangle  $ABC$ . The triangles  $A'B'C'$  and  $A'_1B'_1C'_1$  are respectively its axonometric and secondary projections in the rectangular isometric projection.

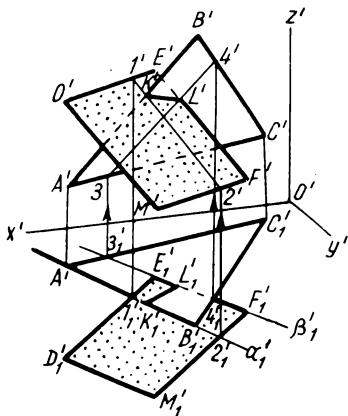


Fig. 385

The construction of the line of intersection  $K'L'$  of triangle  $A'B'C'$  with the parallelogram  $D'E'F'M'$  in the rectangular dimetric projection is shown in Fig. 385. Just as in the multiview drawing the construction is carried out with the help of auxiliary section planes. The point of intersection  $K'$  of the side  $A'B'$  of the triangle with the plane of the parallelogram is constructed by using the horizontal-projecting plane  $\alpha$ , drawn through side  $A'B'$ . Trace  $\alpha'_1$  coincides with the secondary projection  $A'_1B'_1$ . Point  $K'$  lies at the intersection of line  $A'B'$  and line  $l'2'$  obtained as the result of intersection of plane  $\alpha$  with the plane of the parallelogram.

Point  $L'$  of line  $K'L'$  is obtained with the help of a similar horizontal-projecting plane  $\beta$  passing through one side  $E'F'$  of the parallelogram. Point  $L'$  is given by the intersection of the section line  $3'4'$  and line  $E'F'$ .

The visibility of portions of the intersecting figures in the axonometric representation is established, as in the multiview drawing, by comparing the coordinates of separate points (see § 23, Fig. 193).

Let us consider some examples of the construction of axonometric projections of various geometrical figures and details such as encountered in engineering drawing.

1. The construction of the isometric projection of a cube (Fig. 386) is started by drawing the axis  $O'z'$ . At an angle of  $120^\circ$

to  $O'z'$ , or at an angle of  $30^\circ$  to the horizontal, the axes  $O'x'$  and  $O'y'$  are then drawn, this is stage *a* seen in Fig. 386.

Then we construct the projection  $A'B'C'D'$  of the top face of the cube  $ABCD$  (stage *b*). Along axis  $O'x'$  from point  $O'$  we lay off  $O'B' = A'B'$  equal to 0.82 of the length of the edge  $AB$ . Along axis  $O'y'$  from point  $O'$  we lay off  $O'D' = A'D' = A'B'$ .

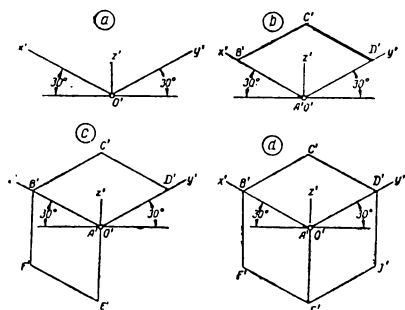


Fig. 386

Through the points  $B'$  and  $D'$  obtained we draw lines parallel to axes  $O'y'$  and  $O'x'$ . These lines intersect in point  $C'$ . The rhomb  $A'B'C'D'$  is the isometric projection of the top face  $ABCD$  of the cube.

Let us construct the projection  $A'B'F'E'$  of the face  $ABFE$  lying in the plane  $z'O'x'$  (stage *c*). For this purpose we lay off downwards from point  $A'$ , along axis  $O'z'$  the foreshortened length of the edge of the cube. The factor is 0.82. From point  $B'$  parallel to  $O'z'$  we lay off  $B'F' = A'E'$ . We then join point  $E'$  with points  $A'$  and  $F'$ , and point  $F'$  with point  $B'$ . The rhomb  $A'B'F'E'$  is the isometric projection of the lateral face  $ABFE$  of the cube.

In a similar manner we construct the projection  $A'D'J'E'$  of the second lateral face  $ADJE$  of the cube (stage *d*).

The hidden edge of the cube is not shown. In axonometric representations hidden outlines are generally excluded in order not to make the drawing too complicated. The faces of the cube in its isometric projection appear as rhombs, which are all the same in form and size. The outline of the complete projection of the cube is a regular hexagon. The axonometric axes and factors of distortion along these axes may be shown apart from the axonometric projection as, for example, in Figs. 388 and 389.

2. In order to avoid the large number of arithmetical computations involved in measuring the length of the various segments of straight lines and multiplying these numbers by the distortion factor (in this case 0.82), an angular scale may be used (Fig. 387). Such a scale is based on the proportional division of lines of different lengths, which form an angle with each other.

To construct the required scale a horizontal line 100 mm long is drawn. Another line 82 mm long is drawn to form any convenient angle with the first line. The ratio of the length of these lines is  $82 : 100 = 0.82$ . Having divided the horizontal line into 10 equal segments or parts and denoting the points of division as 10, 20, etc., we join the extreme points 100 and 82 by a straight line. We then

pass a series of lines parallel to line 100-82 through the points of division. These parallels divide the second line into 10 equal parts. The length of each segment received will be 00.82 of the length of each segment of the horizontal line.

In order to determine the length of the axonometric projection of any line a segment of the same length is first laid off from point

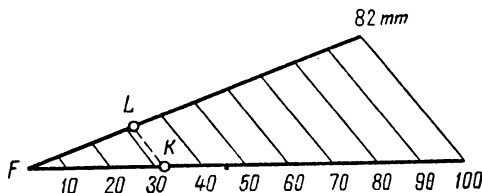


Fig. 387

$F$  on the horizontal line. Then through the point  $K$  obtained a line  $KL$  is drawn parallel to lines of the scale. The segment  $FL$  gives the required length of the isometric projection of the original line. Angular scales may be constructed to give any degree of foreshortening, including, of course the dimetric factors of distortion 0.94 and 0.47.

3. The dimetric projections of a cube are built in the same sequence as isometric projections. In Fig. 388, *a* a cube in the dimetric projection is shown. The factor of distortion or foreshortening along axis  $y'$  is 0.47.

Distortion factors are often rounded off by using a factor equal to unity instead of 0.94 along axes  $O'x'$  and  $O'z'$  and along axis  $O'y'$  0.5 instead of 0.47.

When rounded off distortion factors are used the representation is called a *reduced pictorial* representation (Fig. 388, *b*). If the lengths of the edges of the cube, seen in Fig. 388, *a*, are compared with those of the edges of the cube in Fig. 388, *b*, we find that the dimetric representation of the cube seen in Fig. 388, *b* is drawn to a larger scale, i. e., 1 : 0.94, or 1.06 : 1.

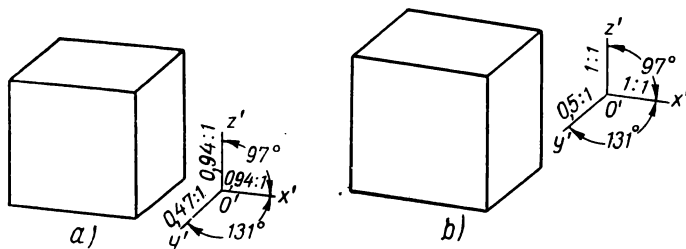


Fig. 388

In the dimetric projection of a cube two of its faces, the front and rear, parallel to the coordinate plane  $xOz$  are rhombs with angles  $83^\circ$  and  $97^\circ$ . All other faces are parallelograms with angles of  $49^\circ$  and  $131^\circ$ .

4. When the face of an object shown in axonometric projection is an irregular polygon, it is advisable to inscribe this polygon

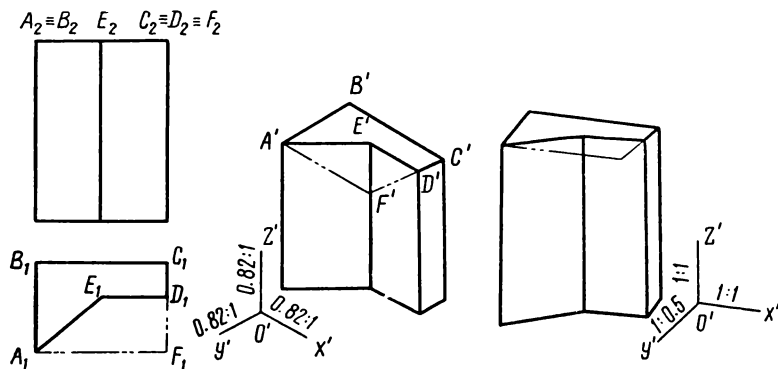


Fig. 389.

within a rectangle, the sides of which are parallel to the coordinate axes. This method is used when constructing the representations of a prism in isometric and dimetric projection (Fig. 389). After constructing the projection  $A'B'C'F'$  of the rectangle  $ABCF$  we find the axonometric projections  $E'$  and  $D'$  of points  $E$  and  $D$  by means of their coordinates  $x$  and  $y$ . Then from points  $A'$ ,  $B'$ ,  $C'$ ,  $D'$  and  $E'$  we drop straight lines parallel to axis  $O'z'$ . On these lines we lay off the length of the lateral edges taking into consideration the distortion factor along axis  $O'z'$ . The points thus obtained are joined by straight lines.

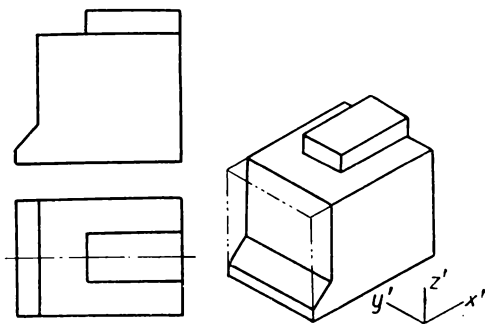


Fig. 390

5. The isometric projection of the object shown in Fig. 390 is constructed in the following sequence. First, a parallelepiped is drawn, then its left part is eliminated, and finally the top shoulder is drawn in.

6. The axonometric projection of a polyhedron, such as the irregular

pyramid, shown in Fig. 391, should be built by using the coordinates of its vertices. The construction is simplified due to the fact that the vertices  $A$ ,  $B$  and  $C$  of the triangular base of the pyramid and the foot, or lower point  $D$ , of the line of altitude, are all contained in the coordinate plane  $xOy$ . Having constructed the isometric axes  $O'x'$ ,  $O'y'$  and  $O'z'$ , we then find the axonometric

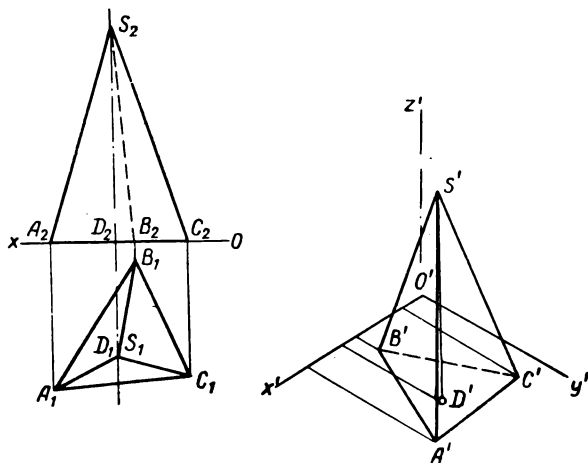


Fig. 391

projections  $A'$ ,  $B'$  and  $C'$  of the points of the triangular base. For this purpose, using the multiview drawing, we determine the coordinates of the points  $A$ ,  $B$  and  $C$ . Then, using the angular scale (Fig. 387), and by means of the values of  $x$  for points  $A$ ,  $B$  and  $C$ , we find the axonometric values  $x'$  of these points. These segments are then marked off from point  $O'$  along axis  $O'x'$ . From the points obtained on axis  $O'x'$  we lay off on lines parallel to axis  $O'y'$  the values  $y'$  for the vertices. The latter are determined by measuring the values  $y$  for points  $A$ ,  $B$  and  $C$  in the multiview drawing and reducing them by means of the corresponding angular scale. Points  $A'$ ,  $B'$  and  $C'$  are joined by straight lines. The triangle  $A'B'C'$  is the isometric projection of the base of the pyramid  $ABC$ .

Point  $D'$ , the isometric projection of the foot of the line of altitude  $SD$ , is constructed in a similar way. Through point  $D'$  upwards we draw a vertical and lay off on it the foreshortened altitude of the pyramid (the factor is 0.82). This gives us point  $S'$ , i. e., the isometric projection of the pyramid vertex  $S$ . We finish the construction of the isometric representation of the pyramid by joining  $S'$  with points  $A'$ ,  $B'$  and  $C'$  by straight lines.

7. When the outline of the lower part of the object is more complicated than its upper part, it is advisable to begin the

axonometric by choosing a point of sight so that the object is viewed from below. This corresponds to the case when the eye of the observer is in one of the octants 3, 4, 7 or 8. In Fig. 392 an example of such a construction in dimetric projection is shown. The direction of the axes  $x'$ ,  $y'$  and  $z'$  indicate that the observer's eye is in the 8 octant.

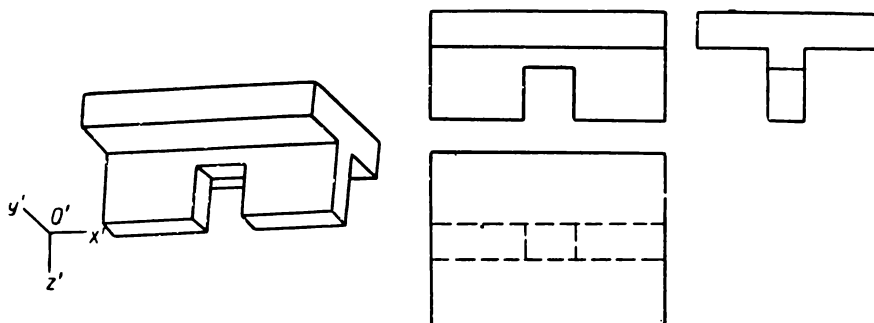


Fig. 392

8. It should be noted that the axonometric projection of an object may be also constructed directly in the multiview drawing by rotating the object successively about two axes or by projecting it onto new planes of projection.

The construction of the isometric and the dimetric projections of a cube is shown in Figs. 393 and 394. The isometric projection of the cube was obtained by substituting the planes of projection  $\Pi_1$  and  $\Pi_2$  by new planes  $\Pi_4$  and  $\Pi_5$ . The dimetric projection of a cube was obtained by successive rotating of the cube about two axes. The first axis coincides with one of the lateral edges of the cube and the second passes through the left vertex of the base and is perpendicular to plane  $\Pi_2$ .

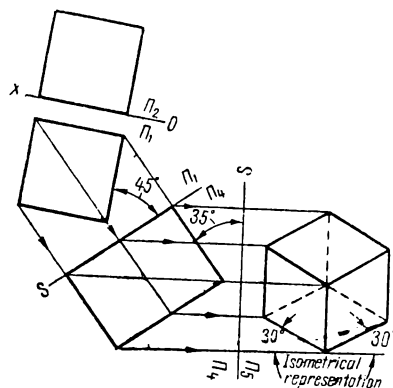


Fig. 393

9. The circles inscribed into the cube faces in isometric projection become ellipses inscribed in rhombs which are themselves the isometric projections of the faces of the cube (Fig. 395).

The centres of these ellipses lie at the intersection of the diagonals of the rhombs. The major diagonals  $B'D'$ ,  $B'E'$  and  $E'D'$  of the rhombs are respectively perpendicular to axes  $O'z'$ ,  $O'x'$  and  $O'y'$  and are equal to the true lengths of the diagonals

of the cube (see Fig. 385). It follows that the major axes of the ellipses  $1'-2'$ ,  $5'-6'$  and  $9'-10'$ , which coincide with these diagonals, are of the same length as the diameters of the circles inscribed in the faces of the cube.

The minor axes  $3'-4'$ ,  $7'-8'$  and  $11'-12'$  of the ellipses coincide with the minor diagonals of the rhombs and are perpendicular to

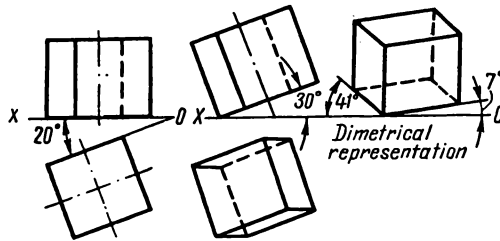


Fig. 394

the major axes of the ellipses. The minor axes of the ellipses are the isometric projections of lines, each of which is perpendicular to the two edges of the cube, which it intersects. In the isometric projection each axis lies on the same straight line as do these two edges. So the direction of the minor axis of any of the ellipses is parallel to the axonometric axis, that is perpendicular to the plane in which the given circle lies.

The minor axes of the ellipses are equal in length to 0.57 of the diameters of the circles inscribed in the faces of the cube. This may be readily seen if we consider the right triangle  $S'B'C'$  (Fig. 395), where  $C'S' = B'S' \cdot \tan 30^\circ$ . It follows that  $2C'S' = 2B'S' \cdot \tan 30^\circ$ , or  $A'C' = B'D' \cdot \tan 30^\circ$  and  $A'C' = 0.57B'D'$ . Since axis  $3'-4'$  is perpendicular to  $B'D'$ , then the same ratio 0.57 exists between the minor axis of the ellipse and diameter of the circle inscribed in the face of the cube.

It may be seen from Fig. 395 that in each of the three ellipses, besides four points which are the extreme points of the axes of the ellipse, there are also four points where the sides of the rhomb are tangential to the ellipse. These points  $K'$ ,  $L'$ ,  $M'$ ,  $N'$ ,  $P'$ ,  $Q'$ ,  $T'$ ,  $W'$  and  $R'$  are located on lines parallel to the isometric axes and are projections of the points of contact

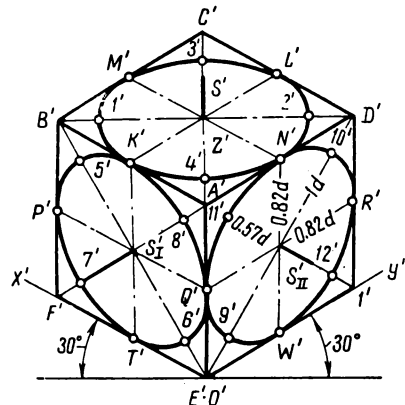


Fig. 395



of the edges of the cube which are tangential to the circles inscribed in the various faces.

**10.** The construction of the ellipse having its centre in point  $S'$ , which is the isometric projection of the circle inscribed in the top face of the cube, is shown in Fig. 396. First a line  $S'A'$  is drawn

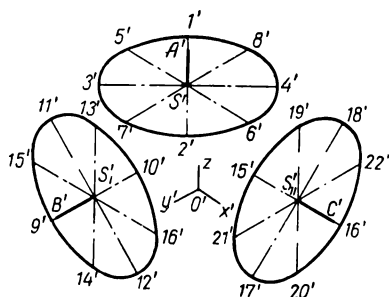


Fig. 396

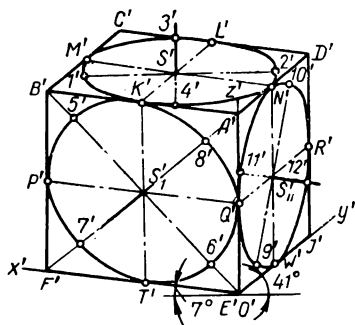


Fig. 397

parallel to the axis  $O'z'$ . This line determines the direction of the minor axis of the ellipse  $1'-2'$ . The length of this minor axis is equal to  $0.57d$ , where  $d$  is the diameter of the circle. The major axis  $3'-4'$  of the ellipse is perpendicular to the axis  $1'-2'$  and the length of this major axis is equal to  $d$ . Further, through centre  $S'$  of the ellipse lines  $5'-6'$  and  $7'-8'$  are drawn respectively parallel to axes  $O'x'$  and  $O'y'$ . The length of each of these lines is equal to  $0.82d$ . The ellipse is then drawn through the eight points obtained.

The other two ellipses, with centres in points  $S'_I$  and  $S'_{II}$  which are the isometric projections of the circles in the coordinate planes  $zOx$  and  $zOy$ , are constructed in a similar way. The major axis of the left ellipse is perpendicular to axis  $O'y'$  and the major axis of the right ellipse is perpendicular to axis  $O'x'$ .

When a circle does not lie in a coordinate plane or in a plane parallel to a coordinate plane, the ellipse may be constructed by using the coordinates of a series of points taken on this circle.

**11.** Isometric projections are often constructed without using the factor of distortion 0.82 to foreshorten dimensions along the axes  $Ox'$ ,  $Oy'$  and  $Oz'$ . In this case the lengths of the axes of the ellipses — the isometric representation of the circles lying in the coordinate planes, or in planes parallel to them, — are determined as follows, for the major axis,  $1:0.82=1.22$  of the diameter of the circle and for the minor axis,  $0.57:0.82=0.7$  of the diameter of the circle.

12. The dimetric projection of a cube with circles inscribed in its faces is shown in Fig. 397. The front and rear faces of the cube are seen as rhombs and the other faces as parallelograms. The minor axes of the ellipses are parallel to the respective dimetric axes, whereas the major axes are perpendicular to the minor axes.

The lengths of the major and minor axes of the ellipse depend on whether the circle lies in a top, lateral or front face, and also on the scale to which the dimetric projection is drawn. If the dimetric projection of the cube is foreshortened using the factor 0.94 in the direction of the axes  $O'x'$  and  $O'z'$  and 0.47 in the direction of axis  $O'y'$ , then the major axis of each of the ellipses is equal to the diameter of the circle itself. If the dimetric projection is constructed without foreshortening along axes  $O'x'$  and  $O'z'$ , but is foreshortened along axis  $O'y'$ , using the factor 0.5, then the major axis of each of the three ellipses is equal to  $1:0.94=1.06$  of the diameter. The minor axis of an ellipse, located in the coordinate plane  $z'O'x'$ , is equal to 0.88 of the length of the major axis. The minor axis of ellipses located in planes  $x'O'y'$  and  $z'O'y'$  is equal to  $\frac{1}{3}$  of the length of the major axis.

The ratio of the lengths of the major and minor axes of the ellipses,  $1:0.88$ , may be easily obtained if we consider the right triangle  $F'S'E'$ , where  $S'F' = S'E' \cdot \tan 41^\circ 25' = 0.88 S'E'$ . It follows that  $A'F' = 0.88 B'E'$ . This ratio also exists between axes  $7'-8'$  and  $5'-6'$  of the ellipse in this coordinate plane  $x'O'z'$ . Therefore, if the length of the major axis  $5'-6'$  of the ellipse is assumed equal to diameter  $d$  of the circle represented, then the length of the minor axis  $7'-8'$  is equal to  $0.88 d$ . If the length of the major axis  $5'-6'$  is assumed as equal to  $1.06 d$ , then the length of the minor axis is equal to  $0.88 \times 1.06 d = 0.93 d$ .

The relationship between  $O'K$  and  $OK$ , i. e., between the axonometric projection  $O'K$  of the line and the line  $OK$  itself (Fig. 398), may be considered as the relationship between the minor and major axes of the ellipse located in the coordinate plane  $x'O'y'$  (Fig. 397). As can be seen from the drawing in Fig. 398,  $\frac{O'K}{OK} = \sin \gamma^\circ = \sqrt{1 - \cos^2 \gamma^\circ}$ . The value  $\cos \gamma^\circ$  is the factor of distortion  $r$  along axis  $O'z'$ , but since  $r = \frac{2\sqrt{2}}{3}$ , it follows that  $\frac{O'K}{OK} = \sqrt{1 - \frac{8}{9}} = \frac{1}{3}$ .

Besides the extreme points of the axes for each ellipse in the dimetric projection of a cube (Fig. 397), we have another four points:  $K', L', M'$  and  $N'$  for the upper ellipse,  $N', W', Q'$  and  $R'$  for the lateral ellipse, and  $K', T', P'$  and  $Q'$  for the front ellipse. Each of these points is the mid-point of the corresponding side of the rhomb or parallelogram and represents the dimetric projection of the point in which the circle touches the edges of the cube.

13. The construction of the ellipses, which are the dimetric projections of circles lying in the coordinate planes  $zOx$ ,  $xOy$  and  $zOy$ , is shown in Fig. 399. To construct the top ellipse we take a point  $S'$  as its centre and then draw a line  $S'A'$  parallel to axis  $O'z'$ . Assuming this as the line of the minor axis we draw, at right angle to it, the line of the major axis of the ellipse. The length of

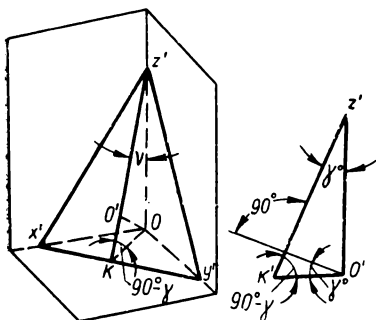


Fig. 398

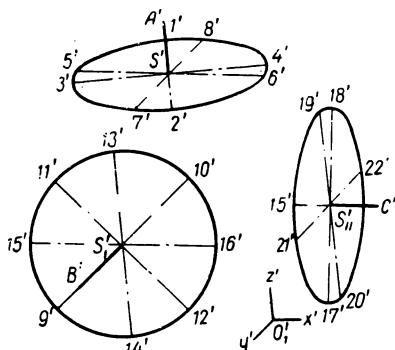


Fig. 399

the major axis is taken as equal to 1.06 of the diameter of the circle, the scale being 1.06:1. By laying off along the major axis, to both sides of point  $S'$ , 0.53 of the diameter of the circle we obtain points  $3'$  and  $4'$ . Next, by laying off from point  $S'$  upwards and downwards along axis  $O'z'$   $1/6$  of the length of the major axis we obtain the extreme points of the minor axis of the ellipse, i. e., points  $1'$  and  $2'$ . Then, through that same point  $S'$ , we draw lines respectively parallel to axes  $O'x'$  and  $O'y'$ . On the first line, at a distance from point  $S'$  equal to the radius of the circle, we obtain points  $5'$  and  $6'$ . On the second line at a distance from point  $S'$  equal to one half of the radius of the circle we obtain points  $7'$ ,  $8'$ . The upper ellipse is drawn through the eight points found in this way.

The ellipse with its centre in point  $S''_{II}$  is constructed in a similar manner.

When constructing the ellipse with its centre  $S'_I$  on the line  $S'B'$ , which determines the direction of the minor axis, we lay off not  $1/3$ , but 0.88 of the length of the major axis of the ellipse, and both  $13'-14'$  and  $15'-16'$  are taken as being equal to the diameter of the circle.

14. Now let us consider the construction of the isometric and dimetric projections of a cylinder, a cone and a sphere.

After constructing the isometric projection of the top of the cylinder (Fig. 400), from the extreme points of the major axis of the ellipse we draw two vertical lines and lay off on them downwards segments equal to 0.82 of the altitude of the cylinder. After this we draw one half of the lower ellipse which is the isometric projection of the base of the cylinder. To construct the half of the lower ellipse on the front part of the top ellipse we take a series of points and drop verticals from them. On these verticals we mark off 0.82 of the height of the cylinder. The half of the lower ellipse is obtained by drawing a curve through the points obtained.

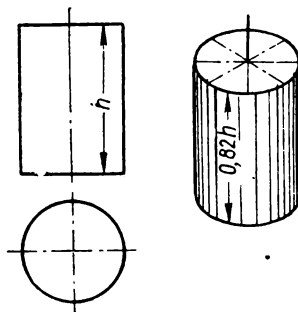


Fig. 400

The isometric projection of a cylinder with its axis in a horizontal position (Fig. 401) is begun by constructing the right-hand end. In this case the ellipse is drawn with the help of chords of the circle. One of the diameters parallel to one of the coordinate axes, say  $Oy$ , is divided into a number of parts and through each of the points obtained chords perpendicular to the axis are drawn. Having constructed the axonometric projection  $A'B'$  of this diameter  $AB$ , we divide it into the same number of parts and draw through the points of division a series of straight lines in the direction of the chord of the circle, in this case parallel to axis  $O'z'$ . On these lines, to both sides of the isometric projection of the diameter, we lay off 0.82 of the length of the corresponding semi-chords of the circle. The ellipse is then drawn through the points obtained.

The outline generatrices of the cylinder and the visible part of the left end of the cylinder should be drawn as shown in Fig. 400.

15. An example of the construction of a dimetric projection of a detail, consisting of two vertical cylinders with a through cylindrical bore, is given in Fig. 402. In the multiview drawing in accordance with the rules of working sections in engineering drawings, half of the view is shown alongside half of the vertical section.

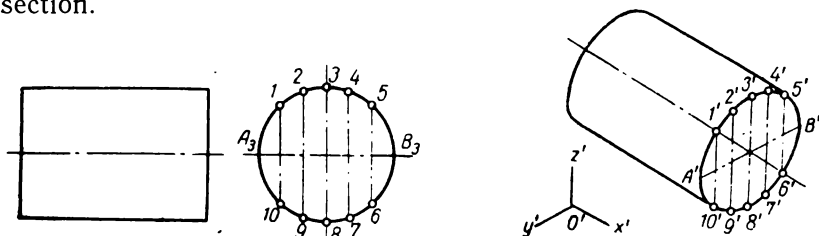


Fig. 401

In order to show the complete form of the detail in the dimetric projection part of it has been removed.

We first construct the dimetric projection of the top of the lower cylinder by means of chords. In Fig. 402 only the construction of points 1 and 2 is shown, these being the extreme points of the chord passing through point A. The dimetric projection  $1'-2'$  of this chord, parallel to axis  $O'y'$ , has been reduced to half the size of

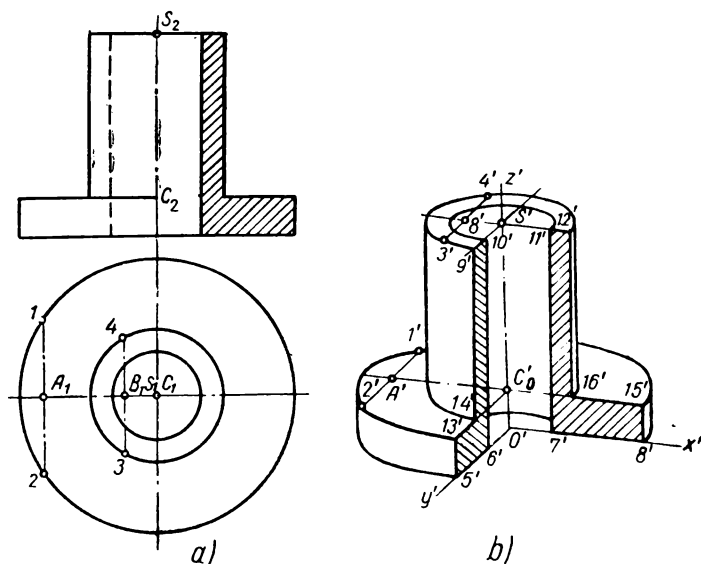


Fig. 402

the chord 1-2. Having constructed the ellipse, which is the dimetric projection of the top of the lower cylinder, we draw in the outline generatrices of this cylinder. Next we draw in the visible part of the ellipse which is the dimetric projection of the base of cylinder.

The construction of the dimetric projection of the top cylinder is begun by constructing the upper face. To find the dimetric projection of the centre  $S'$  of this ellipse, from point  $C'$ , which is the centre of the ellipse which was constructed first, we lay off the length of the generatrix of the upper cylinder. The ellipse with its centre at point  $S'$  is also constructed with the help of chords. In Fig. 402 the construction of points 3' and 4' is shown. The ellipse representing the upper circle of the bore is constructed in the same manner. The centre of this ellipse is again point  $S'$ .

Having constructed the dimetric projection of the detail, we eliminate the right-hand front part cut by the coordinate planes  $x'O'z'$  and  $y'O'z'$ . Points 5' and 8' are given by the intersection of the lower ellipse with axes  $O'y'$  and  $O'x'$ . Points 9', 10', 11' and 12'

are given by the intersection of the upper ellipses with lines drawn through their common centre  $S'$  parallel to axes  $O'y'$  and  $O'x'$  respectively. Lines parallel to the corresponding dimetric axes should be drawn through the points obtained and through points  $13'$  and  $14'$ ,  $15'$  and  $16'$  which were found in the same way. This gives us the outlines of the sections cut by the coordinate planes  $x'O'z'$  and  $y'O'z'$ .

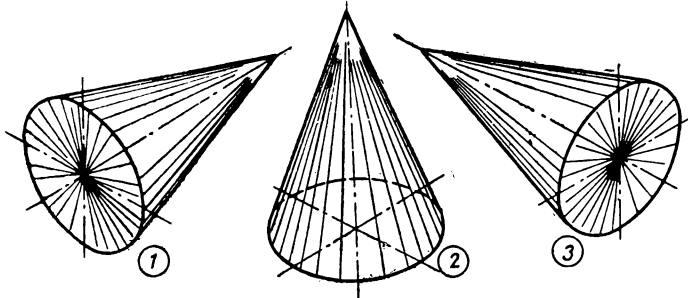


Fig. 403

After the removal of a portion of the detail, part of the base of the cylindrical bore becomes visible. To construct the visible arc of this ellipse several points are taken on the corresponding part of the upper inside ellipse. Verticals are dropped from these points. On them we mark off the altitude of the cylindrical bore. The ellipse is then drawn through the points obtained.

16. Three isometric projections of a right circular cone (its axis in each case parallel to one of the coordinate axes) are given in

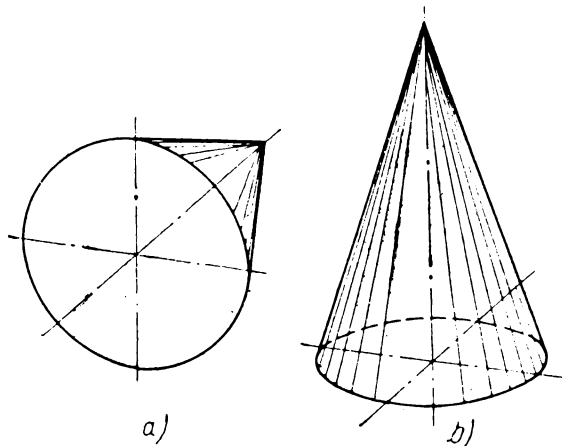


Fig. 404

Fig. 403. In the first case the axis of the cone lies in the direction of the axis  $Oy$ . In the second case its direction is that of axis  $Oz$  and in the third, that of axis  $Ox$ .

The dimetric projections of a cone are shown in Fig. 404. In the first case (Fig. 404, *a*) the base of the cone lies in a plane

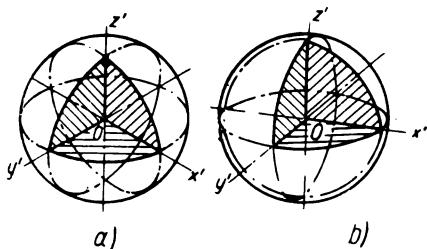


Fig. 405

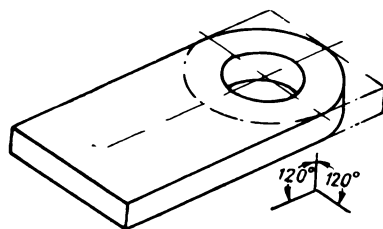


Fig. 406

perpendicular to the axis  $Oy$ , and in the second case (Fig. 404, *b*) it is perpendicular to the axis  $Oz$ . In both cases the drawing of the cone is fulfilled without foreshortening along axes  $Ox$  and  $Oz$ , but it is foreshortened along axis  $Oy$ , the factor being 0.5. Therefore the major axis of the ellipses, i. e., the dimetric projections of the base, is equal to 1.06 of the diameter of the circle. The minor axes of these ellipses are respectively equal in length to 0.88 and  $\frac{1}{3}$  of the major axis. The ellipses themselves are drawn through eight points.

17. In rectangular isometric projections a sphere appears as a circle whose diameter is equal to the diameter of the sphere. In Fig. 405, *a* the isometric projection of a sphere 22 mm in diameter is constructed with foreshortening along axes  $O'x'$ ,  $O'y'$  and  $O'z'$ , the factor being 0.82. On the isometric projection of the sphere three ellipses are shown. These are the isometric projections of the three

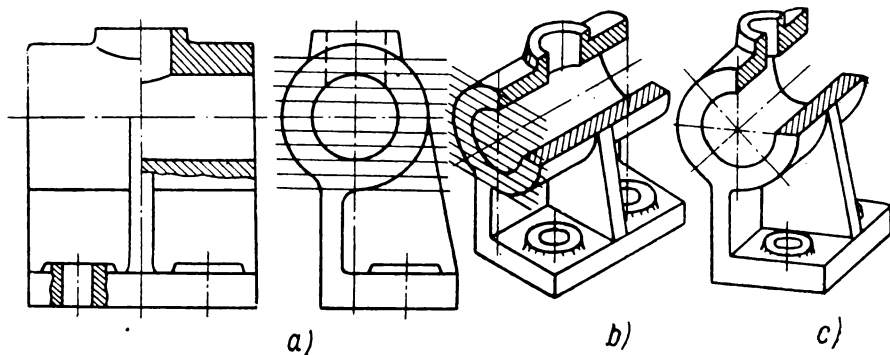


Fig. 407

major circles of a sphere located in the coordinate planes. In the drawing one eighth of the sphere is removed.

In rectangular or right dimetric projections the sphere also appears as a circle (Fig. 405, *b*). The diameter of this circle will be 1.06 of the diameter of the sphere, if the drawing is constructed

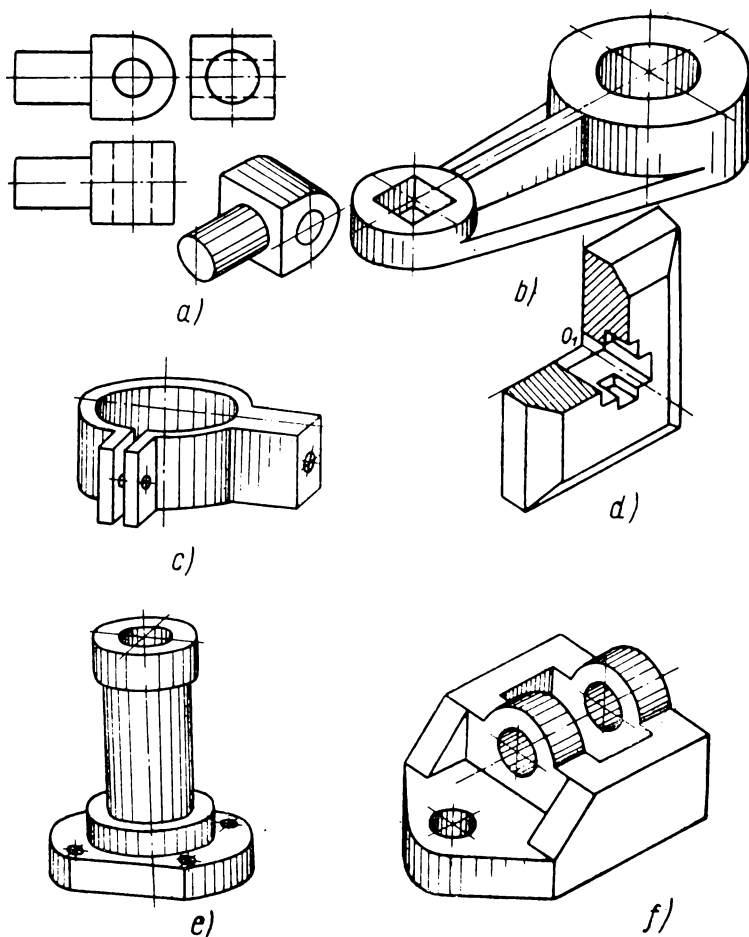


Fig. 408

without foreshortening along axes  $O'x'$  and  $O'z'$ , but with foreshortening along axis  $O'y'$ , the factor being 0.5. On the dimetric projection of a sphere three ellipses are shown. These are the projections of the three major circles of the sphere and they lie in the coordinate planes. One eighth of the sphere is removed.



18. If an object with a rounded end is to be represented in axonometric projection, as in Fig. 406, the outlines of the object are first constructed without any rounding off as is shown by broken lines. Only after that is the half of the ellipse, representing the rounded end, constructed.

The construction of the bracket (Fig. 407, *a*) both in isometric projection (Fig. 407, *b*) and dimetric projection (Fig. 407, *c*), should be begun by drawing its base which appears as a parallelepiped. The position of the axis of the cylindrical bush is determined next. Then the bush itself is drawn in. After this the edges and lugs with cylindrical bores are constructed. Finally,  $\frac{1}{4}$  of the bush is removed. In Figs. 407, *b* and 407, *c* all the ellipses were constructed by means of chords.

Some details drawn in rectangular axonometric projection are shown in Fig. 408. These details are a lug (isometry, Fig. 408, *a*), a connecting rod (isometry, Fig. 408, *b*), a collar (dimetry, Fig. 408, *c*), a base plate with a section removed (isometry, Fig. 408, *d*), a king-post (dimetry, Fig. 408, *e*) and the body of a jig (isometry, Fig. 408, *f*).

## § 46. Oblique Axonometric Projections

1. Oblique axonometric projections may be classified as isometric, dimetric and trimetric, depending on the position in space of the planes of axonometric projection and the direction of the projectors. The most widely used oblique projections are those in which the plane of axonometric projection  $\Pi'$  is parallel to the coordinate plane  $xOz$  or parallel to the horizontal coordinate plane  $xOy$ .

When plane  $\Pi'$  is in such a position the distortion factor along two of the axonometric axes is unity. Along the third axis, depending on the direction of projection, we may obtain the same distortion factor, or any other. In the first case the oblique axonometric projection is isometric, in the second—dimetric. The axonometric representation obtained when the factor of distortion is greater than unity is not employed since it is distorted.

Axonometric projections, in which the plane of projection  $\Pi'$  is parallel to one of the coordinate planes  $xOz$  or  $xOy$ , are advantageous because all lines, angles and figures lying in planes, parallel to  $\Pi'$ , are projected onto the latter without distortion.

In such a case a condition is that the direction of projection should not be perpendicular to plane  $\Pi'$ . This follows from the definition of the oblique axonometric projection itself. Moreover, the direction of projection should not be parallel to the third coordinate plane  $yOz$ , otherwise the axis  $O'y'$  would become

perpendicular to axis  $O'x'$ , and the representation would not be sufficiently pictorial (Fig. 409).

2. The value of the distortion factor along axis  $O'y'$  and the angle between this axis and the axis  $O'x'$  may be chosen arbitrarily. In Fig. 410 the following elements are shown: the plane of projection  $\Pi'$ , the projection of the point of origin of coordinates — point  $O'$ , and the projection  $O'y'$  of axis  $Oy$ .

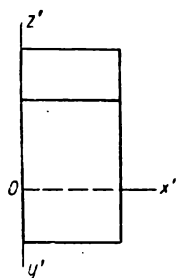


Fig. 409

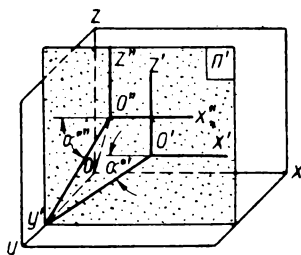


Fig. 410

If triangle  $OO'y'$  is revolved about axis  $Oy$ , the side  $Oy'$  remains stationary, and the vertex  $O'$  will move in plane  $\Pi'$  along an arc of a circle with its centre in point  $y'$  and whose radius is  $O'y'$ . In point  $O''$  we have a new position of the vertex  $O'$  of the triangle. Since  $O'y' = O''y'$ , the distortion factor along axis  $y'$  remains the same ( $O'y' : Oy' = O''y' : Oy'$ ). The direction of the axonometric projection, when point  $O'$  moves into position  $O''$ , changes. The angle between axis  $O'y'$  and the extension of axis  $O'x'$  also changes, see angles  $\alpha'$  and  $\alpha''$  in Fig. 410.

Leaving  $O'y'$  stationary, we can transfer the origin of the axonometric axes  $O'$  to a new position  $O_2''$  on the axis  $O'y'$  (Fig. 411). In this case the direction of the projection changes, but the angle  $\alpha'$ , between axis  $O_2''y'$  and the extension of the axonometric axis  $O''x''$  remains equal to the angle  $\alpha''$  formed by the same axis  $O'y'$  and extension of axis  $O'x'$ . The distortion factors  $O'y' : Oy'$  and  $O''y' : Oy'$  also change.

From the analyses given above it follows that in oblique axonometric projections the angle between the axis  $O'y'$  and the extension of axis  $O'x'$  may be changed, as may the factor of distortion along axis  $O'y'$ . In other words, we may select both of these values at random. In each case the combination will correspond to a definite direction of the oblique axonometric projection.

This basic concept of the theory of axonometric projection was first put forward by Prof. Polke in 1853 in his theorem: Three lines of any length, lying in one plane and emerging from one point at any angles to each other,

represent a parallel projection of three lines of equal length laid off on rectangular axes of coordinates from point of origin of the coordinates  $O$ .

This theorem was later proved for the more general case of central projection by Prof. Chetveroukhin, Prof. Glagolev and other Soviet mathematicians.

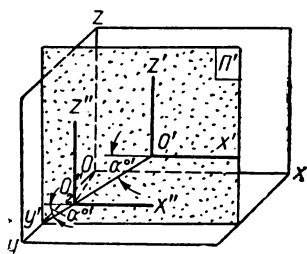


Fig. 411

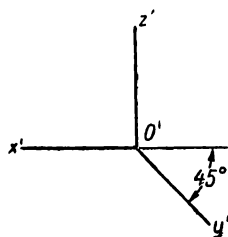


Fig. 412

Axonometric oblique projections obtained in plane  $\Pi'$  parallel to plane  $zOx$  are called *frontal projections*. The angle  $\alpha'$  between axis  $O'y'$  and the extension of axis  $O'x'$  in these projections is assumed as  $30^\circ$ ,  $45^\circ$ , or  $60^\circ$ . When angle  $\alpha' = 45^\circ$ , and the distortion factor along axis  $O'y'$  is  $1/2$ , these projections are called *cabinet oblique projections* (Fig. 412).

When plane  $\Pi'$  is horizontal, and the object is viewed from above, the oblique axonometric projections obtained are called *military projections*.

Three frontal projections of a cube with the angle  $\alpha$  equal to  $45^\circ$ ,  $60^\circ$  and  $30^\circ$ , and the distortion factor equal to  $1/2$  along axis  $O'y'$  are shown in Fig. 413, a.

Three frontal isometric projections of the same cube with a distortion factor equal to unity along all three axes are shown in Fig. 413, b. In Fig. 414 a cube is shown in oblique axonometric projection. The angles of inclination of axes  $O'x'$  and  $O'y'$  to the horizontal are both  $45^\circ$ .

A cube, with circles inscribed in its faces, is shown in frontal projection in Fig. 415. To construct the ellipses — the projections

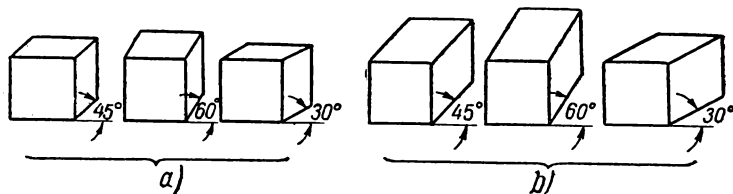


Fig. 413

of the circles — chords are drawn. For example, to construct the points  $1'$  and  $2'$  of the upper ellipse, through any point  $K'$ , taken on line  $C'K'$ , which is parallel to axis  $O'y'$ , we draw a horizontal line. Point  $K$  should then be found on the horizontal projection of the cube, Fig. 415, *b*. The distortion factor along axis  $O'y'$  is 0.5, so  $CK = 2C'K'$ . Through  $K_1$  we draw a chord and transfer the distances  $K_1J_1$  and  $K_1Z_1$  to the axonometric representation.

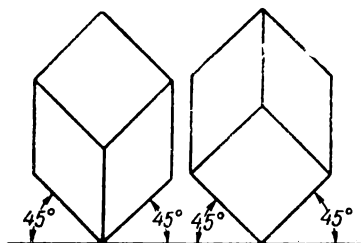


Fig. 414

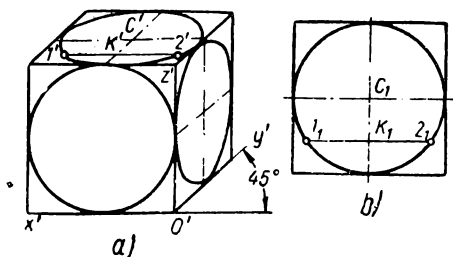


Fig. 415

It should be kept in mind that the major axis of the top ellipse in the frontal axonometric projections is not horizontal. Similarly in the lateral face the major axis of the ellipse will not be perpendicular to axis  $O'x'$ .

On the front face the inscribed circle in the frontal projection remains the same as in the orthographic projection. This is why frontal projections are the most convenient forms of pictorial representations when the axis of a body of revolution being depicted is perpendicular to the frontal plane of projection  $\Pi_2$ .

3. Oblique axonometric projections are constructed by the same methods as used for rectangular axonometric projections.

For example, let us examine the construction of an oblique axonometric projection of the line of intersection of two cylinders (Fig. 416). This construction may be carried out with the help of auxiliary cutting planes. In this case the cutting planes chosen are parallel to plane  $\Pi_3$ . These planes cut both cylinders along their generatrices. The points contained in the line of intersection are given by the intersection of the respective generatrices in one and the same cutting plane. Point  $K$  of this line, for example, is found at the intersection of generatrices lying in the plane passing through the axes of the cylinders. The points of intersection, given

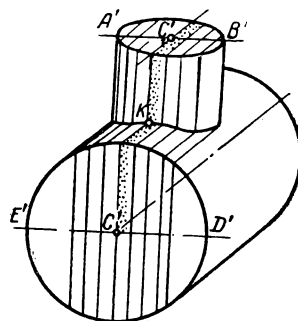


Fig. 416

by other auxiliary planes parallel to the axes of the cylinder, are also points contained in the line of intersection of the given cylinders.

### Problems and Exercises

1. How are axonometric projections classified?
2. What is a trace triangle?
3. What are the values of the factors of distortion in isometric rectangular projections?

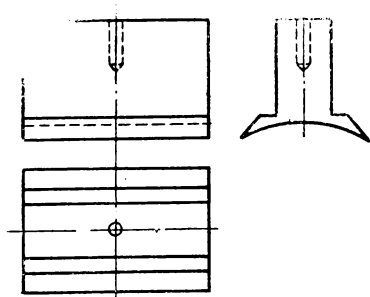


Fig. 417

4. How is an angular scale constructed for a foreshortening of 0.47 in dimetric projection?

5. Explain the method of constructing the ellipses which are the isometric projections of circles in the coordinate planes.

6. What are the dimensions of the major and minor axes of an ellipse, which is the projection of a circle contained in plane  $xOy$ , if the isometric projection is constructed a) with a foreshortening

of 0.82 along axes  $Ox$ ,  $Oy$  and  $Oz$ , b) without any foreshortening along these axes?

7. What are the dimensions of the major and minor axes of the ellipse, which is the projection of a circle contained in plane  $zOx$ , if the dimetric projection is constructed a) without any distortion along the axes  $Ox$  and  $Oz$  and with a distortion factor of 0.5 along axis  $Oy$ , b) with a distortion factor of 0.94 along axes  $Ox$  and  $Oz$  and 0.47 along axis  $Oy$ ?

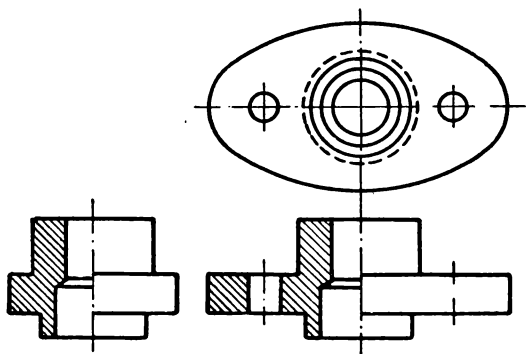


Fig. 418

8. Which axonometric projections are called a) frontal, b) cabinet and c) military?

9. Construct the rectangular isometric projection of the magnet core, shown in Fig. 417. There should be no foreshortening along the axes.

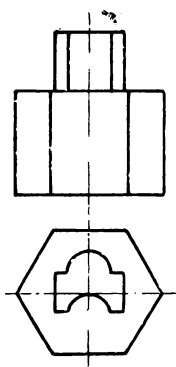


Fig. 419

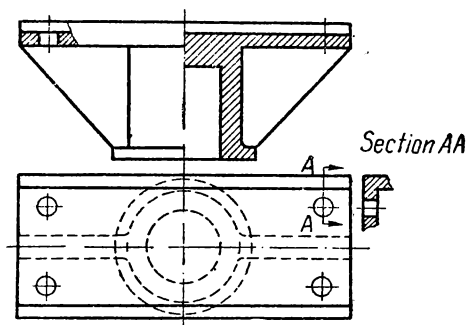


Fig. 420

10. Construct the rectangular isometric projection of the gland cover, shown in Fig. 418, with no foreshortening along the axes.

11. Construct the isometric (or dimetric) projection of a sphere, diameter  $D=60$  mm. One quarter of the sphere should be removed.

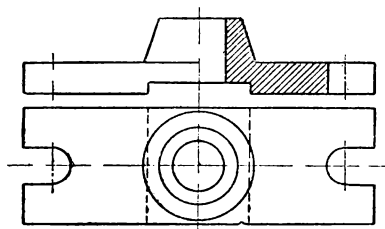


Fig. 421

12. Construct the frontal axonometric projection ( $\alpha=45^\circ$ ) of the fixture, shown in Fig. 419.

13. Construct the rectangular dimetric projection of the bracket, shown in Fig. 420.

14. Construct the rectangular isometric projection of the cross socket, shown in Fig. 421.

## Chapter XII

### CURVED LINES AND CURVED SURFACES

#### § 47. Curved Lines

1. There are two general classes of curved lines, plane curves and space curves, often referred to as single-curved and double-curved lines.

The construction of the second projection of a plane curve, having one of its given projections, is possible only if two projections of a plane in which this curve is contained are given. The construction of the horizontal projection of a curve  $ABCDE$  lying in the plane of triangle  $TMN$ , by means of its frontal projection, is shown in Fig. 422.

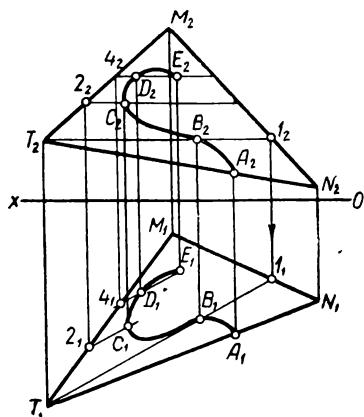


Fig. 422

The construction consists in drawing horizontals of the plane of the triangle through points  $B, C, D$  and  $E$  of the curve. The horizontal projections  $B_1, C_1, D_1$  and  $E_1$  of these points are found on the horizontal projections of the respective horizontals. The horizontal projection  $A_1$  of point  $A$  is determined directly, since point  $A$  belongs to the side  $TN$  of the triangle.

A continuous curve, the horizontal projection of curve  $ABCDE$ , is drawn through points  $A_1, B_1, C_1, D_1$  and  $E_1$ .

2. The construction of the curve  $ABCDEF$  in the oblique plane  $\alpha$  is shown in Fig. 423, *a*. First, plane  $\alpha$  is made to coincide with one of the planes of projection, say with plane  $\Pi_1$ . The curve  $\overline{ABCDEF}$  is then constructed in the new position of plane  $\alpha$ . Horizontals of plane  $\alpha$  are drawn through points  $\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}$  and  $\overline{F}$ . With the help of these horizontals we find the horizontal projections

$A_1, B_1, C_1, D_1, E_1, F_1$  and the frontal projections  $A_2, B_2, C_2, D_2, E_2, F_2$  of the points of the given curve. Through the corresponding projections of points  $A, B, C, D, E$  and  $F$  we draw the projections of the curve lying in the oblique plane  $\alpha$ .

3. To determine the length of a plane curve the plane in which the curve lies is brought into coincidence with one of the planes of projection. The curve in this position is then divided into small parts. Each part is assumed to be a segment of a straight line. These segments are then laid off on a straight line in the same order as on the curve. The sum of all the segments is approximately equal to the length of the curve. This method is used to find the length of the curve  $ABCDEF$ , as shown in Fig. 423, *b*, and is often referred to as the development of the curve.

4. When making the multiview drawing of a space curve it is often necessary to letter certain of its points in order to avoid confusion when reading the drawing if different parts of the projections of the curve lie on the one and same projectors. For example, if points  $K$  and  $L$  were not denoted on the projections of curve  $ABC$  (Fig. 424, *a*) it would not be clear which of the two parts  $AB$  or  $BC$  is nearer to plane  $\Pi_2$ .

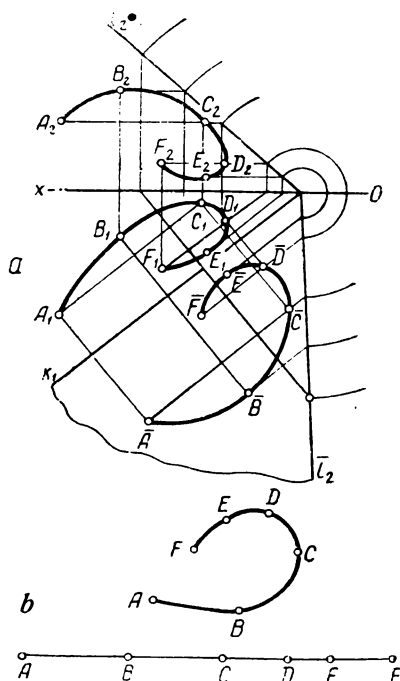


Fig. 423

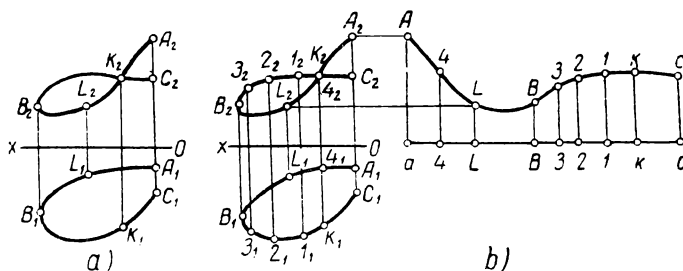


Fig. 424



5. In order to determine the approximate length of a space curve one of its projections, say the horizontal projection, is divided into parts, as seen in Fig. 424, *b*. These segments are marked off successively on a straight line. Through the points of division perpendiculars to the line are erected. On these perpendiculars the distances, at which the frontal projections of the corresponding points lie from the axis  $Ox$ , are marked. A smooth curve is then drawn through these points. The space curve is said to have been "unrolled" or developed into a plane curve.

## § 48. Cylindrical and Conical Helical Lines

1. Helical lines are the most widely used space curves in engineering practice.

A *cylindrical helical line* or *helix* is a spiral in space, contained in the surface of a circular cylinder and intersecting all its generatrices at the same angle. The helix may be considered as being generated by a point contained in the surface of a right circular cylinder and simultaneously rotating about the axis of the cylinder as it moves parallel to the axis. The longitudinal movement is assumed to be proportional to the angular movement.

The axis and diameter of the helix are the axis and diameter of the cylinder in which the helix lies. On the surface of one and the same cylinder different helices may be obtained by varying the ratio of the two velocities of the generating point. In this case the pitch of each of the helices generated on the cylinder will differ from that of the others. The pitch or lead  $h$  of a

helix is the distance between the two nearest points in which the helix intersects one of the generatrices of the cylinder. In other words, the pitch is the distance the point moves parallel to the axis as it completes one revolution.

The part of a helix between the nearest points on the generator, namely the part generated in one revolution, is generally called a *turn* of the helix.

A circular cylinder on which a helix has been plotted is shown in Fig. 425.

Since all the points of the cylindrical helix are located on the lateral surface of the cylinder, which is projected onto plane  $\Pi_1$  as a circle, it is obvious that the horizontal projections of all points of the helical line also lie on this circle.

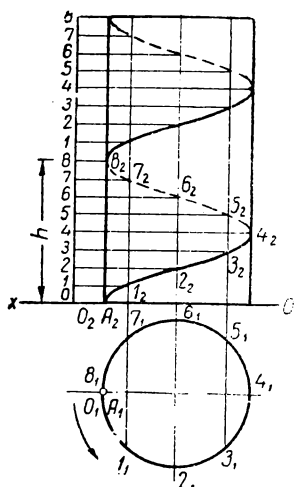


Fig. 425

In order to construct the frontal projection of a cylindrical helix we divide the circular base of the cylinder into a number of equal parts, in this case into eight parts. A definite value must be taken as the pitch  $h$  of the helix. The pitch is then divided into the same number of equal parts as the base, since the linear movement of point  $A$  is proportional to its angular movement.

After point  $A$  has moved from its initial position  $O$  to position  $I$ , i. e., after it has moved through one eighth of the pitch  $h$ , its horizontal projection will occupy position  $I_1$ , and its frontal projection will be  $I_2$ .

A horizontal line is drawn through the first division of the line of pitch. Then, through the first division  $I_1$  of the circle a vertical is passed. The intersection of these two lines determines the position of the frontal projection  $I_2$  of point  $A$  in its new position. The other points,  $2_2, 3_2, \dots, 8_2$ , of the frontal projection of the helical line are constructed in a similar manner.

The frontal projection of the first turn of the cylindrical helix is obtained by drawing a curve through the points  $O_2, I_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2$  and  $8_2$ . The frontal projection of the second turn is constructed in a similar manner.

Helices may be right-hand or left-hand, depending on which direction the point generating the helix moves. Right-hand helices, in other words helical lines with a right-hand thread, are those generated by the clockwise rotation of a point about an axis when the point moves along the axis away from the observer. Left-hand helices are generated by counter-clockwise rotation.

A helix with a right-hand thread is shown in Fig. 425. One of the characteristics of such a helix is that the visible parts of the curve rise to the right and the hidden parts rise to the left. The characteristic feature of a helix with a left-hand thread is that the visible parts of the curve rise to the left and the hidden parts rise to the right.

A helix is also obtained on the cylinder if a right triangle  $A_0B_0C_0$  is wrapped around a cylinder (Fig. 426). The horizontal side  $A_0B_0$  of the triangle is equal to the length of the circle of the base of the cylinder. The vertical side  $B_0C_0$  is equal to the pitch  $h$  of the helix. It follows that the hypotenuse  $A_0C_0$  of the right triangle  $A_0B_0C_0$  is the development of one turn of the cylindrical helix. The angle  $\alpha$  is called the *lead angle*, or the *helix angle*. The size of this angle depends on the pitch  $h$  of the helix and length of the circumference of the base of the cylinder. We have:  $\tan \alpha^\circ = \frac{h}{2\pi R}$  where

$R$  is the radius of the circumference of the cylinder. The length  $L$  of one turn of the helix may be determined from the formula  $L = \sqrt{h^2 + (2\pi R)^2}$ .

The frontal projection of the cylindrical helix is a sine wave. This may easily be proved. Let us assume the axis of projection

as the axis  $x$  and the frontal projection  $J_1J'_2$  of the axis of the helix as the axis  $z$  (Fig. 427). We take any point  $K$  on the helix. The angle of revolution of point  $K$  from its initial position  $O$  is denoted by  $\gamma^\circ$  ( $\angle \gamma^\circ = \angle O_1J_1K_1$ ). The radius of the helix is denoted by  $r$  and the pitch by  $h$ . The coordinates  $x$  and  $z$  of point  $K$  are shown in Fig. 427.

From the right triangle  $J_1N_1K_1$ , we have

$$x = r \sin \gamma^\circ. \quad (1)$$

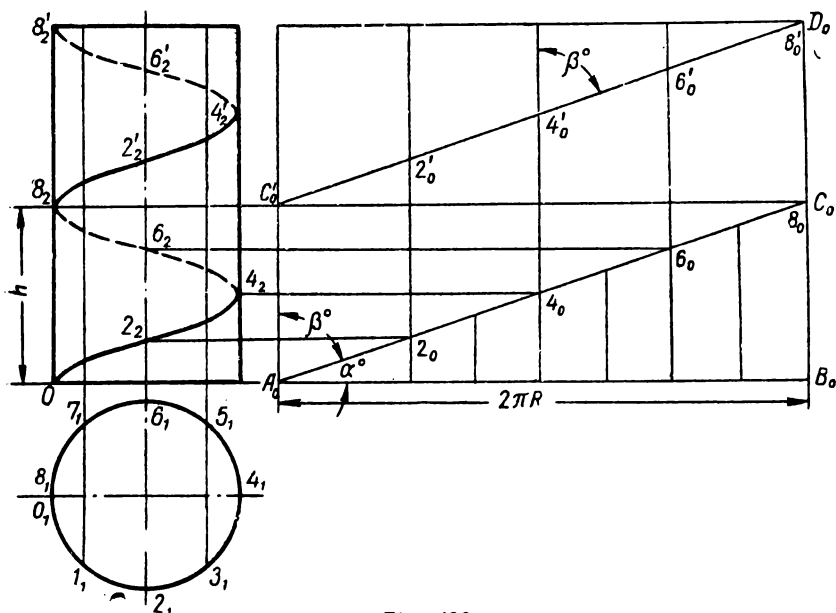


Fig. 426

Since the linear and angular movements of the point forming the helix are proportional, then  $\frac{z}{h} = \frac{\gamma^\circ}{2\pi}$ ,

$$\text{hence } \gamma^\circ = \frac{z \cdot 2\pi}{h} = 2\pi \frac{z}{h}. \quad (2).$$

Substituting the value found for  $\gamma^\circ$  in equation (1), we obtain  $x = r \cdot \sin 2\pi \frac{z}{h}$ , which is the equation of a sine wave.

The angle  $\beta$  on the development of the helix (Fig. 426) is equal to the angle between the generatrix of the cylinder and the helix. This angle  $\beta$  in space is equal to the angle between one of the generatrices of the cylinder and a tangent to the helix.

Since all the generatrices of the cylinder form the same angle  $\beta$  with the helix it follows that all tangents to the helix have the

same slope relative to the generatrices of the cylinder passing through the corresponding tangent points.

The generatrices of the cylinder are perpendicular to plane  $\Pi_1$ , so the slope of all the tangents to the helix, relative to plane  $\Pi_1$ , is the same and equal to the angle  $\alpha$  formed by the hypotenuse and horizontal side of the right triangle  $ABC$ . On the basis of this the tangent to the helix at any point on it can be constructed.

Let us consider the right triangle  $KLM$  (Fig. 427) whose frontal projection  $K_2L_2M_2$  is shown by hatching. This triangle consists of line  $KL$ , a segment of the generatrix of the cylinder passing through the tangent point  $K$ , the tangent  $KM$  to the helix and line  $ML$ . Since the triangle  $KLM$  is located in a horizontal-projecting plane, the line  $ML$ , called the *subtangent*, may be considered as the horizontal projection of the tangent  $KM$ .

It is obvious that angle  $KML = \alpha^\circ$ , i. e., the angle between the tangent to the helix and plane  $\Pi_1$ .

From the right triangle  $KLM$  we have  $ML = KL \cdot \cot \alpha^\circ$ , where  $\alpha^\circ$  is the angle between the hypotenuse  $KM$  and side  $ML$ . But  $OL = KL \cdot \cot \alpha^\circ$ . Therefore  $ML = OL$ , i. e., the length of the subtangent is equal to the length of the straightened arc of the circle — the horizontal projection of the helix — from the initial point  $O_1$  up to the horizontal projection  $K_1$  of the tangent point.

The subtangent  $ML$  is the horizontal projection of the tangent. The tangent is the hypotenuse and the subtangent is the base of the right triangle  $KLM$ . Therefore, the length of the tangent is equal to the length of the straightened arc of the helix itself from the initial point to the point of tangency.

Using these properties of the tangent and subtangent of a helix these lines may be constructed. In order to construct a tangent to the helix through point  $K$  (Fig. 427) we draw a tangent to the circle which is the horizontal projection of the helix. The tangent point is  $K$ , the horizontal projection of point  $K_1$ . Then from point  $K_1$  we lay off  $K_1M_1$ , equal to the length of the straightened arc of the circle from point  $O_1$  to point  $L_1$ . Having obtained point  $M_1$  we find point  $M_2$  by drawing a projector. Point  $M$ , whose projections

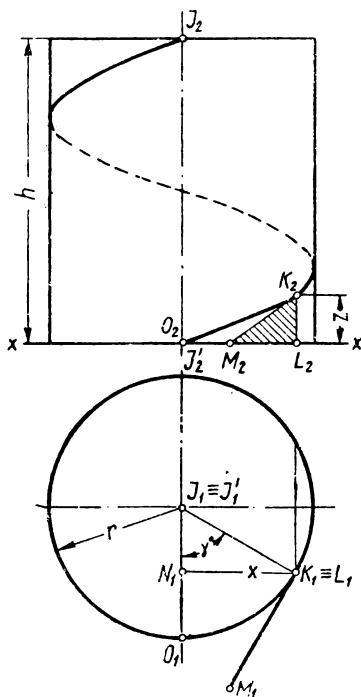


Fig. 427

$M_1$  and  $M_2$  have been found, is the horizontal trace of the required tangent. Joining points  $M_2$  and  $K_2$  by a straight line we obtain the frontal projection  $K_2M_2$  of the tangent  $KM$  to the helix at point  $K$ .

If we revolve triangle  $KML$  about axis  $KL$  into the frontal position, the true value of angle  $KML = \alpha^\circ$  will be found in the multiview drawing. The lead angle  $\alpha$  of the helix is determined by the development of the helix (Fig. 426). The angle  $\alpha$  is formed by the hypotenuse and horizontal side of triangle  $ABC$ .

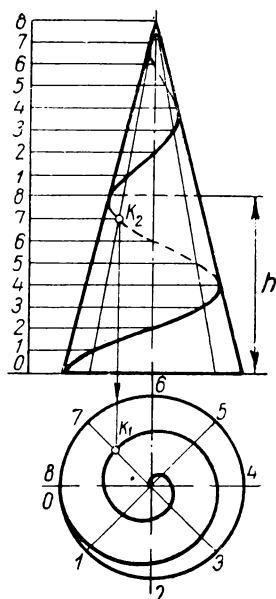


Fig. 428

2. The conical helix (Fig. 428) is generated by a point moving along the surface of a right circular cone so that its movement along the generatrices of the cone is proportional to the angular movement about the axis of the cone.

The movement of the point along the generatrices of the cone may be considered as composed of a movement along the axis of the cone and a movement in a direction perpendicular to this axis.

Since the two linear movements and the angular movement of the generating point are proportional, the construction of a conical helix may be carried out as follows. The circle of the base of the cone is divided into several equal parts, in this case into eight parts. Then the eight corresponding generatrices of the cone are drawn in and their horizontal and frontal projections are constructed. The pitch  $h$  is then divided into the same number of equal parts. Through each division a horizontal is passed to intersect the frontal projection of the corresponding generatrix. For example, point  $K_2$  lies at the intersection of the horizontal line drawn through division 7 of the pitch and the frontal projection of the generatrix drawn through division 7 of the circle of the base.

The points found in this way are the frontal projections of the points of the helix. By using the frontal projections we can obtain the horizontal projections of these points. To do this projectors are dropped from the frontal projections of the points to intersect the horizontal projections of the respective generatrices of the cone. The construction of point  $K$  is shown in Fig. 428. By passing curves through the corresponding projections of these points we obtain two projections of the conical helix. The horizontal projection of the conical helix is an Archimedean spiral. The frontal projection

is a sine curve, the amplitude of which decreases. The construction of two turns of the R. H. conical helix is shown in Fig. 428. The pitch  $h$  is equal to one half of the altitude of the cone.

#### Problems and Exercises.

1. How is the projection of a plane curved line constructed on a given plane?
2. How is the length of a space curve determined?
3. Define a) a cylindrical helix, b) a conical helix.
4. How are the projections constructed a) for a cylindrical helix, b) for a conical helix?
5. Construct a L. H. cylindrical helix, the diameter of which is 40 mm and the pitch  $h=30$  mm.
6. Construct a L. H. conical helix, given the diameter of the base circle of the cone as 50 mm, the altitude of the cone 80 mm, and the pitch  $h=40$  mm.

### § 49. Curved Surfaces

1. Curved surfaces are generated by lines moving in accordance with definite laws. Depending on the form of these lines, or generatrices, curved surfaces are divided into two major classes, single-curved surfaces, or ruled surfaces, generated by moving straight lines, and double-curved surfaces, the generatrices of which are curves. Cylinders and cones are examples of single-curved surfaces.

A sphere is an example of a double-curved surface. The stationary line along which the generatrix of the curved surface moves is called the *directrix*. When a plane, tangent to the surface, passes through all the points contained in each of the generatrices of a single-curved surface consecutively, the single-curved surface is said to be *developable*, i. e., it can be “unwrapped”. When this condition is not fulfilled the single-curved surface is called a *warped surface*. It cannot be unwrapped, so it is not developable. Summing up, a developable surface may be “unwrapped”, so that it coincides with a plane and then has no folds or breaks. Warped surfaces and double-curved surfaces cannot be developed.

2. Cylindrical surfaces. A cylindrical surface is generated by a straight line moving along the directrix  $AB$ , so that it always remains parallel to a given line  $MN^*$  (Fig. 429). In this

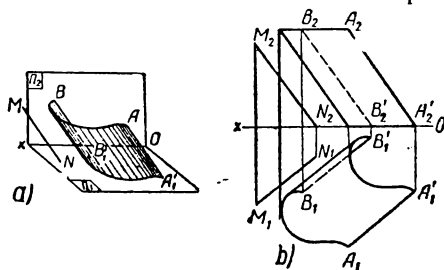


Fig. 429

\* Each position that the generatrix assumes is called an element of the surface.

case the directrix  $AB$  must not lie in the same plane as the generatrix.

The line in which a curved surface intersects a plane of projection is called a *trace* of the surface. In Fig. 429 the curve  $A_1'B_1'$  in plane  $\Pi_1$  is the horizontal trace of the cylindrical surface  $A'B'BA$ .

If a trace of a cylindrical surface\* and the direction of the generatrices are known, the cylindrical surface may be constructed and on it any generatrix may be plotted and any of its points can be determined (Fig. 430).

Of all the generatrices of the cylindrical surface the

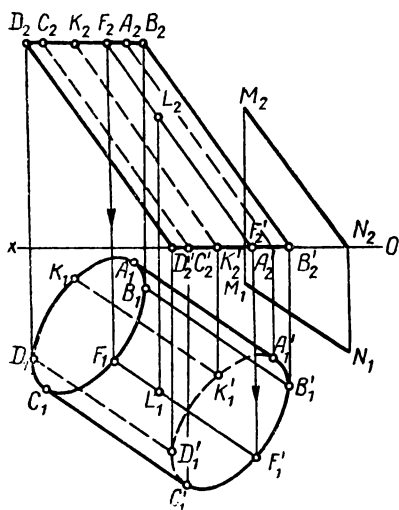


Fig. 430

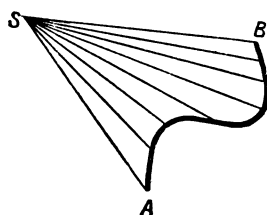


Fig. 431

outline generatrices,  $AA'$  and  $CC'$ ,  $BB'$  and  $DD'$ , are of particular significance.

In the multiview drawing the projections of two outline generatrices are the boundaries of the visible and hidden parts of the surface. In Fig. 430 that part of the cylinder passing through the segment  $A'B'C'$  of the horizontal trace, when projected onto plane  $\Pi_1$ , is visible. The rest of the surface is hidden. When projected onto plane  $\Pi_2$  that part of the cylinder surface passing through the segment  $B'C'D'$  of the horizontal trace is visible. The rest of the surface is hidden.

**3. Conical surface.** The surface generated by the movement of a straight line  $SA$  along curve  $AB$ , when the generating line in all its positions passes through a point  $S$ , is called a *conical surface* (Fig. 431).

Line  $SA$  is the *generatrix*. Point  $S$  is called the *vertex*. Curve  $AB$

\* The trace of the surface is at the same time the directrix of this surface..

is the *directrix*. The conical surface consists of two parts called the *nappes* of the surface (Fig. 432).

If the vertex of the cone and its directrix, or the trace of the cone surface, are known, then the conical surface may be constructed. On the surface any generatrix may be drawn and any point determined. For example, to construct the frontal projection  $K_2$  of point  $K$  given on the surface of the cone (Fig. 433), the horizontal projection  $K_1$  of the point being known, it is sufficient to draw through point  $K_1$  the horizontal projection  $S_1L_1$  of the generatrix  $SL$  of the cone and then find the frontal projection of the generatrix  $S_2L_2$ , after which, from point  $K_1$ , a projector is drawn to intersect  $S_2L_2$  in point  $K_2$ .

In Fig. 433 the lines  $S_2A_2$  and  $S_2B_2$  and also  $S_1C_1$  and  $S_1D_1$  are the projections of the outline generatrices. These generatrices separate the visible and hidden parts of the projections of the conical surface.

4. The developable helicoid. The surface formed by the continuous movement of a straight line, which in all its positions remains tangent to a cylindrical helix, is called a *developable helicoid*, or *helical convolute* (Fig. 434). The cylindrical helix, which in this case is the directrix of the developable helicoid, is the *helical directrix of tangency*. So the developable helicoid belongs to that group of curved surfaces which have a directrix of tangency.

Any section of a developable helicoid given by a plane perpendicular to the axis of a cylindrical helix is an *involute of a circle*.

Part of a helicoid cut by a cylindrical surface whose axis coincides with the axis of a helicoid is shown in Fig. 435.

The line of section in this case will be a helix with the same axis as the helicoid. This surface is called a *developable annular helicoid*.

5. Cyliindroid. The surface generated by a straight line, which constantly remains parallel to a given plane as it moves along two directrices not located in one plane, is called a *cyliindroid*. The plane to which the generatrix of the cyliindroid remains parallel as it moves is called the *plane of parallelism*, or *plane director*.

A representation of a cyliindroid, generated by the movement of the generatrix  $AB$  along two space curves, the directrices, is given in Fig. 436, *a*. The multiview drawing of this same cyliindroid is shown in Fig. 436, *b*. The straight line  $AB$  in each of its position remains parallel to the given plane  $\alpha$ , which is the plane of parallelism.

6. The hyperbolic paraboloid. This surface, sometimes called a warped plane, is a particular case of a cyliindroid (Fig. 437).



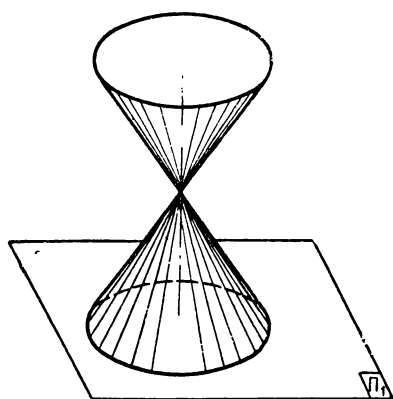


Fig. 432

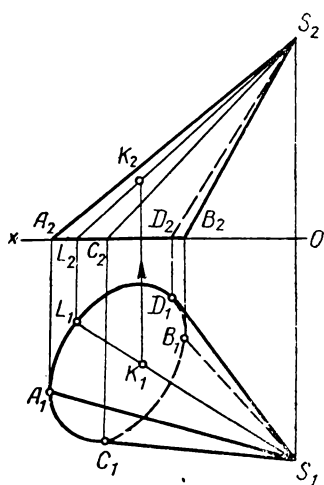


Fig. 433

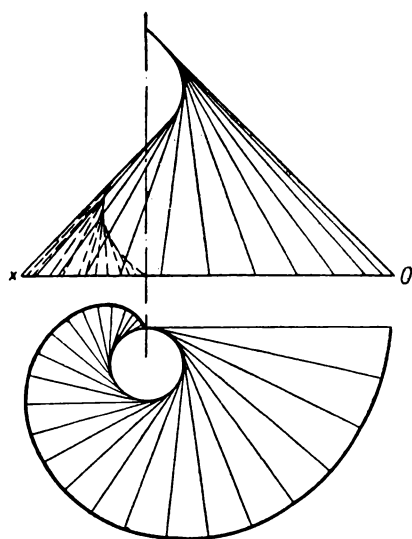


Fig. 434

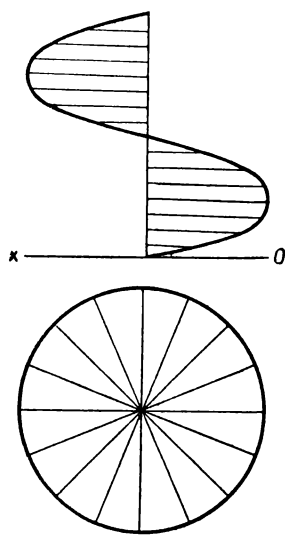


Fig. 435

The directrices of such a surface are two non-intersecting, or skew lines. The horizontal plane of projection  $\Pi_1$  in this case is assumed as the plane of parallelism for the hyperbolic paraboloid (Fig. 437).

Three projections of a hyperbolic paraboloid, having two planes of parallelism  $\alpha$  and  $\beta$ , perpendicular to plane  $\Pi_1$  which form angles of  $\alpha^\circ$  with plane  $\Pi_2$ , are shown in Fig. 438.

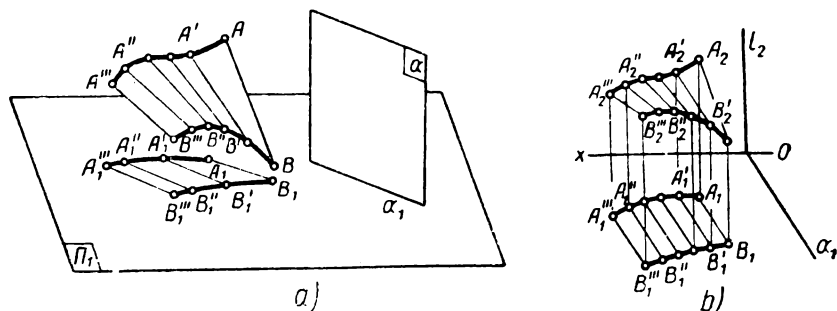


Fig. 436

If we cut such a hyperbolic paraboloid by horizontal planes  $\gamma$ ,  $\gamma'$ ,  $\gamma''$  the sections obtained will be hyperbolas. If such a hyperbolic paraboloid is cut by planes parallel to  $\Pi_2$  or  $\Pi_3$  the sections will be parabolas.

7. The conoid is a particular case of the cylindroid. One of the directrices of the conoid is a straight line, the other is a curve. The generatrix is a straight line which constantly remains parallel to a plane of parallelism.

A conoid is shown in Fig. 439. One directrix of this surface is line  $AA'$ , in this case perpendicular to the plane of parallelism  $\Pi_1$ , the second directrix is curve  $BB'$ .

If the curved directrix of the conoid is a cylindrical helix, the conoid is called a *helical conoid* or *non-developable helicoid* (Fig. 440).

If the helical conoid is cut by the surface of a right circular cylinder, whose axis  $MM'$  coincides with the straight directrix of the conoid, the line of intersection obtained will be a cylindrical helix having the same pitch as the helical directrix  $BB'B''B'''B''''$  of the conoid (Fig. 441). The surface of the helical conoid contained between helical lines is called an *annular helicoid*.

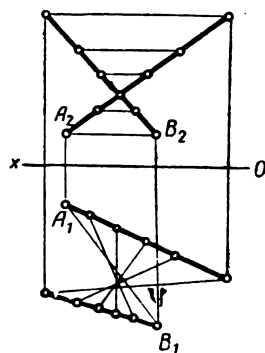


Fig. 437

Annular helical conoids find wide application in engineering practice. Nuts and bolts with rectangular or square threads, cylindrical springs of rectangular section, helical staircases and the Archimedean screw all contain surfaces of helical annular conoids.

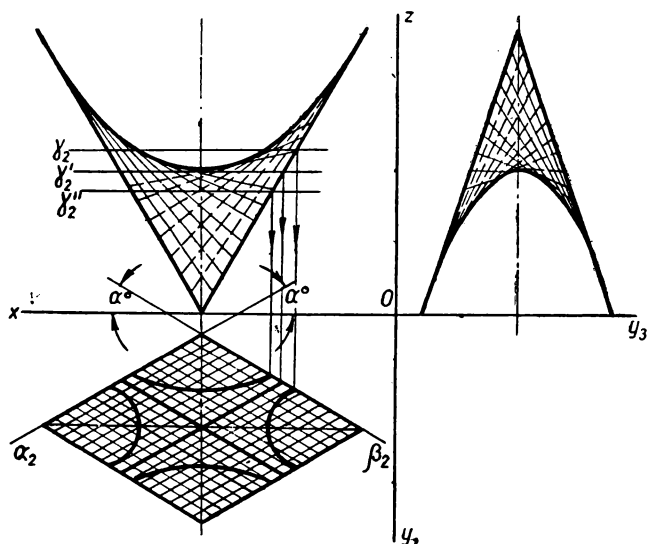


Fig. 438

The points and lines on non-developable, or warped ruled surfaces are constructed the same way as on developable ruled surfaces.

It is required to construct the frontal projection  $K_2$  of a point  $K$

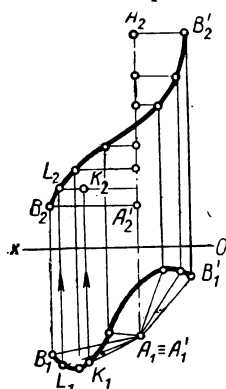


Fig. 439

on the surface of a conoid, the horizontal projection  $K_1$  of this point being known (Fig. 439). Through point  $K_1$  the horizontal projection of generatrix  $AK$  is drawn to intersect the horizontal projection of the curved directrix in point  $L_1$ . From point  $L_1$  we draw a projector to intersect the frontal projection of the curved directrix in point  $L_2$ .

The frontal projection of the generatrix, on which point  $K$  lies, will pass through point  $L_2$  parallel to axis  $Ox$ . A projector is then drawn through  $K_1$  to the frontal projection of the generatrix  $LK$ . In this way frontal projection  $K_2$  of point  $K$  is obtained.

8. The oblique helicoid is one whose generatrix is a straight line that intersects the axis at a constant angle but not at right angles. One point of the generatrix describes a cylindrical helix, while a second point slides along the axis.

If an oblique helicoid is cut by a circular cylinder having the same axis, an *oblique annular helicoid* is obtained (Fig. 442).

The surface of an oblique annular helicoid, which is used for screws with triangular and trapezoid threads, is limited by two cylindrical helical lines of the same pitch. If the generatrices  $1-1'$ ,  $2-2'$ , etc., of this surface are extended, they intersect the axis  $AB$ , making with it the same angle  $\alpha$  in space.

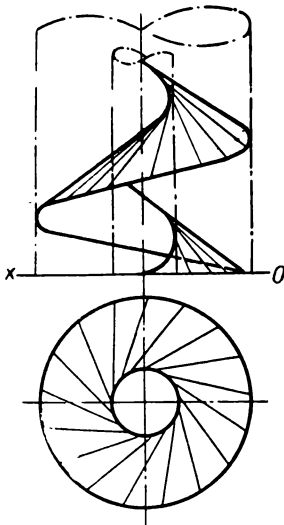


Fig. 440

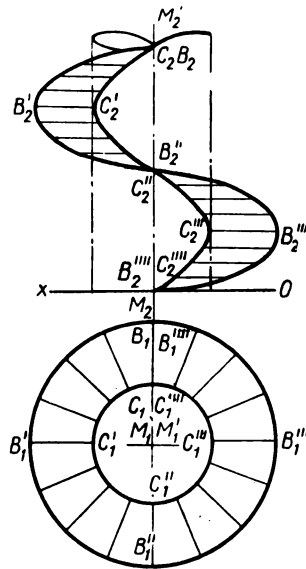


Fig. 441

To construct an oblique annular helicoid we first construct two auxiliary cylinders with a common axis  $AB$ . The circles, which are the horizontal projections of these cylinders, are divided into a number of equal parts, in this case into 12 parts. The division points are joined by straight lines  $1_1-1'_1$ ,  $2_1-2'_1$ , etc. The lines  $1_1-1'_1$ ,  $2_1-2'_1$ , etc., are the horizontal projections of the generatrix of the oblique annular helicoid in its various positions, i. e., of its elements. To find the frontal projections of these lines, we divide the pitch of the helix into the same number of equal parts. These twelve segments are laid off upwards from points  $O_2$  and  $O'_2$ . The latter are the frontal projections of the extreme points of the generatrices which describe helices of the same pitch  $h$  on the surface of the cylinders.

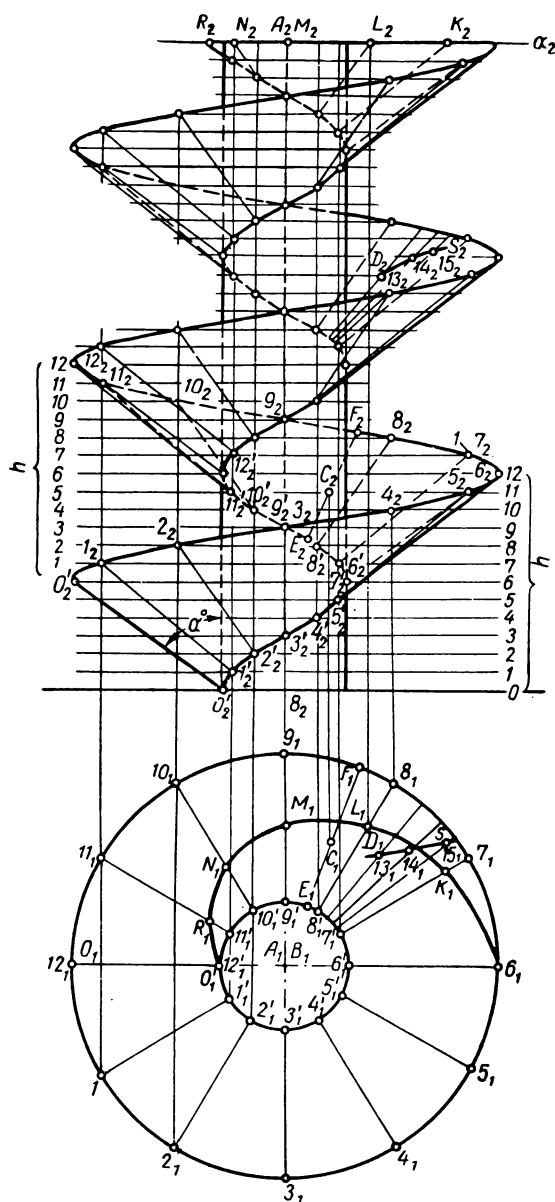


Fig. 442

The line  $O_2O'_2$  is drawn beforehand at the given angle  $\alpha$  to the axis of the helicoid.

Verticals are erected from the division points of the major and minor circles. Horizontals are drawn through the dividing points of the pitch. The intersection of corresponding lines gives us points  $1_2-1'_2, 2_2-2'_2$ , etc. By joining up these pairs of points by straight lines we obtain the frontal projections of the generatrices. Curves are passed through points  $1_2, 2_2, \dots, 12_2$ , contained in the lateral surface of the major cylinder, and through points  $1'_2, 2'_2, \dots, 12'_2$ , contained in the lateral surface of the minor cylinder. These curves are two cylindrical helices and represent the outlines of an oblique annular helicoid.

The construction of the curve  $RMNLK$ , in which the oblique annular helicoid is cut by the horizontal plane  $\alpha$ , consists in determining the points of intersection of a number of elements, or generatrices, with plane  $\alpha$ . The frontal projections of these points are the points of intersection of the frontal projections of the elements with the trace-projection  $a_2$  of plane  $\alpha$ .

The horizontal projections  $R_1, N_1, M_1, L_1, K_1$  of points  $R, N, M, L$  and  $K$  are obtained by dropping projectors from points  $R_2, N_2, M_2, L_2$  and  $K_2$  to intersect the horizontal projections of the respective elements.

The curve  $RMNLK$ , the projection of which on plane  $\Pi_1$  is a true view, is a Spiral of Archimedes.

To construct the frontal projection of any point, such as  $C$ , contained in the surface of the helicoid, given the horizontal projection  $C_1$  of this point, we first draw through point  $C_1$  a radial line  $E_1F_1$ . We then construct the frontal projection  $E_2F_2$  of this element and pass a projector through point  $C_1$  to intersect  $E_2F_2$  in point  $C_2$ .

Let the frontal projection of a curve, such as  $DS$ , contained in the surface of the helicoid, be given. It is required to find the horizontal projection of this curve. On the helicoid, in the region where the line  $DS$  must lie, we draw a series of elements. Then from points  $13_2, 14_2, 15_2$ , in which the frontal projections of the generatrices intersect the given projection  $D_2S_2$  of the curve, we drop projectors to the horizontal projections of the respective elements. This gives us points  $13_1, 14_1$  and  $15_1$  through which a curve is drawn. This curve is the horizontal projection of curve  $DS$  contained in the oblique annular helicoid.

## § 50. Surfaces of Revolution

1. *Surfaces of revolution* are those surfaces that may be generated by revolving a straight or curved line about a stationary axis. Two examples of surfaces of revolution are right circular cylinders and spheres.

The section of a surface of revolution by a plane perpendicular to the axis of revolution is a circle. The most commonly used method of constructing points on surfaces of revolution is based on this fact.

A body of revolution is shown in Fig. 443. Its surface is formed by the revolution of a plane curve  $ABC$  about the axis  $MN$  which

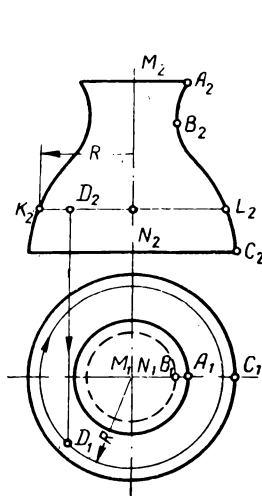


Fig. 443

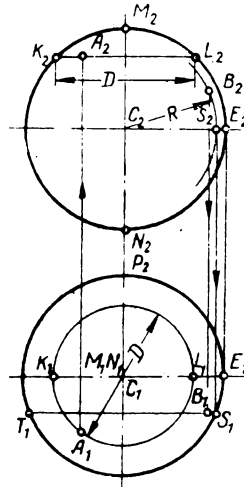


Fig. 444

is perpendicular to plane  $\Pi_1$ . Let us find the horizontal projection of point  $D$  belonging to this surface, if the frontal projection  $D_2$  of point  $D$  is given.

Point  $D$  must lie on a circle, the plane of which is perpendicular to the axis of revolution  $MN$  and, so also perpendicular to plane  $\Pi_2$ . Therefore through point  $D_2$  we draw a horizontal  $K_2L_2$  which is the frontal projection of the required circle. Next, we describe a circle with point  $M_1N_1$  as its centre and radius  $R = \frac{1}{2} K_2L_2$ . This is the horizontal projection of the circle on which point  $D_1$  is found by dropping a projector from point  $D_2$ .

2. Given the horizontal projection  $A_1$  of point  $A$  on the surface of a sphere, Fig. 444, construct its frontal projection. First draw a circle with its centre at  $C_1$  through point  $A_1$ . Having obtained points  $K_1$  and  $L_1$  we then find the frontal projection  $K_2L_2$  of this circle. This will be a line parallel to ground line  $Ox$ . A projector is then drawn through point  $A_1$  to line  $K_2L_2$ . This gives us the frontal projection  $A_2$  of point  $A$  on the surface of the sphere.

In order to construct a point  $B$  on the surface of this same

sphere, given its frontal projection  $B_2$ , first we draw a circle through point  $B_2$  with centre  $C_2$ . Having obtained point  $S_2$  from it we draw a projector to this circle, which is the horizontal projection of the sphere. This gives us point  $S_1$ . Next a line  $S_1T_1$  is passed through point  $S_1$  parallel to line  $Ox$ . This is the horizontal projection of the circle with a radius  $CS$ . On it we find point  $B_1$  with the

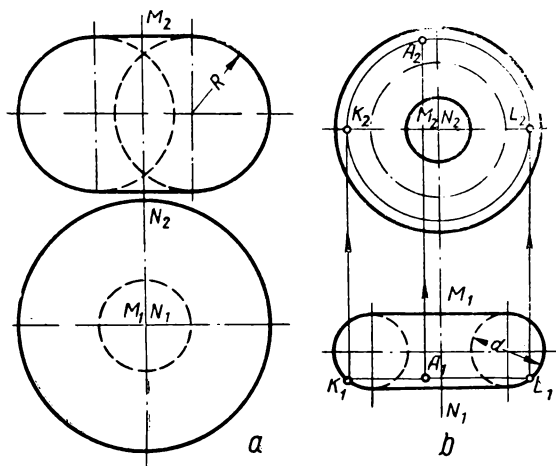


Fig. 445

help of a projector passed through point  $B_2$ . Point  $B_1$  obtained in this way is the horizontal projection of point  $B$  on the surface of the sphere.

It should be noted that the circles  $KAL$  and  $BST$  may be considered as sections of a sphere cut by projecting planes drawn respectively through points  $A$  and  $B$ . Sections of a sphere cut by planes perpendicular to axis  $MN$  (Fig. 444) are called *parallels*. Sections cut by planes passing through the axis  $MN$  of the sphere are called *meridians*. The parallel cut by the plane passing through the centre of the sphere is the *equator*.

The meridian whose plane is parallel to the frontal plane of projection is called the *principal meridian*.

3. The surface represented in Fig. 445,  $a$  is obtained by the revolution of a circle of radius  $R$  about axis  $MN$  which intersects the circle, but does not pass through its centre. If we were to revolve a circle of diameter  $d$  (Fig. 445,  $b$ ) about axis  $MN$ , also located in the plane of the circle, but not cutting it, we would obtain the surface called an *annular torus* or *tore*.

To construct a point on the surface of a torus it is sufficient to draw an auxiliary circle  $KL$  on the surface (Fig. 445,  $b$ ). On the



projections  $K_1L_1$  and  $K_2L_2$  of the circle points  $A_1$  and  $A_2$  are taken. These are the projections of point  $A$  belonging to the surface of the torus. Points on the surface of revolution, shown in Fig. 445,  $a$ , are constructed in a similar manner.

4. The surface formed by the revolution of an ellipse about one of its axes is called an *ellipsoid of revolution*, or a *spheroid* (Fig. 446).

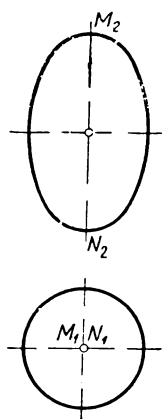


Fig. 446

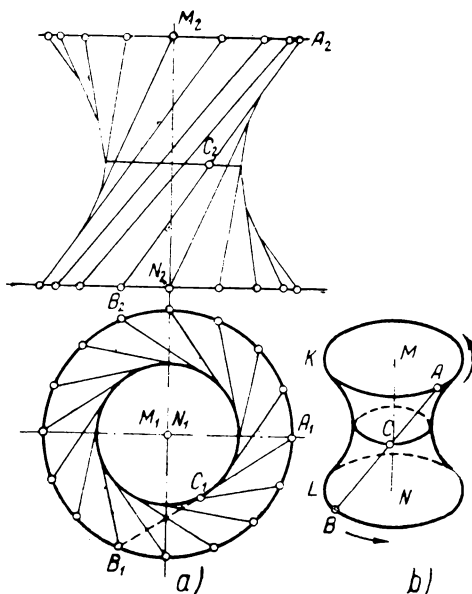


Fig. 447

5. A *hyperboloid of revolution of one nappe* is the surface generated by a line that rotates about the axis, but does not intersect it (Fig. 447).

All points of the generatrix  $AB$  of the hyperboloid of revolution of one nappe describe circles with their centres on axis  $MN$ . The circle described by the point  $C$ , nearest to the axis  $MN$ , is called the *neck*.

A hyperboloid of one nappe may also be obtained by revolving a hyperbola about an imaginary axis. In Fig. 447,  $b$  the imaginary axis is  $MN$  and the generatrix is the hyperbola  $KL$ .

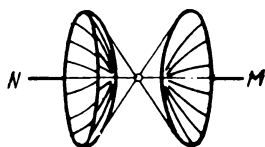


Fig. 448

The section of a hyperboloid of revolution cut by a plane perpendicular to its axis is a circle, and the section cut by a plane passing through the axis of the hyperboloid is a hyperbola.

The surface of revolution we have described is termed a hyperboloid of one

nappe in order to distinguish it from the hyperboloid of two nappes, shown in Fig. 448. The latter is obtained by revolving a hyperbola about its real axis.

The single-nappe hyperboloid of revolution, which is a warped surface, finds many applications in engineering practice, for example, in the design of radio towers and supports for tanks, etc.

6. If a parabola is rotated about its axis the surface obtained is called a *paraboloid of revolution*.

#### Problems and Exercises

1. What is meant by single-curved surfaces and double-curved surfaces? Give examples of such surfaces.
2. Name and define developable single-curved surfaces.
3. What are undevelopable surfaces?
4. Give the definition of the following surfaces: a cylindroid, a hyperbolic paraboloid, a conoid, a helical conoid, an annular helical conoid, an oblique helicoid and an oblique annular helicoid.
5. What are surfaces of revolution? Give examples of these surfaces.
6. What is the usual procedure for constructing points and lines on surfaces?
7. Construct a right L. H. helicoid.
8. Construct an oblique L. H. helicoid.

#### § 51. Cylindrical Screws

1. The surface generated by the movement of a plane figure over the surface of a cylinder along a helix contained in it, so that the plane of the figure always passes through the axis of the cylinder, is called a *helical surface*. The body limited by this surface is called a *screw*. The threads of screws are of different shapes, depending on the form of the generating profile, i. e., on the shape of the plane figure being moved. Among the threads most commonly used are those of square, rectangular, triangular, trapezoidal and semicircular cross sections.

In Fig. 449, *a* the construction of one turn of a triangular thread is shown.

Threads may be right-hand (R. H.) and left-hand (L. H.). When a R. H. thread is generated the generating profile moves counter-clockwise, when the screw is viewed from above. In the drawing this means that the front, visible side of the thread rises from left to right. When a L. H. thread is generated the reverse is true.

When inserting a screw into a nut with a R. H. thread the screw is rotated clockwise. When inserting a screw into a nut with a L. H. thread the screw is turned counter-clockwise.

If the screw is generated by the movement of only one profile, as in Fig. 449, *a*, it is called a *single-thread*, or a *single-cut screw*.

If two or more similar generating profiles are moved, assuming them joined together, *double-thread screws*, *triple-thread screws* and so on are obtained.

When drawing a double-thread screw it is necessary to construct helical lines equal in number to the number of vertices

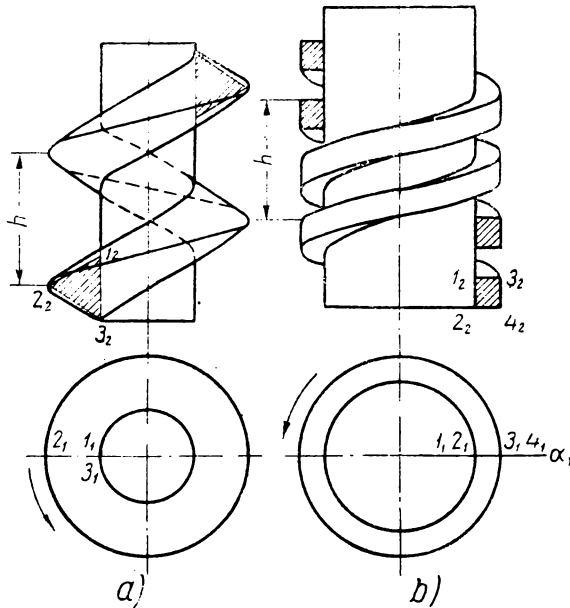


Fig. 449

of the two generating profiles. For example, when constructing a double-thread screw (Fig. 449, *b*) it is necessary to construct eight helical lines, corresponding to the number of vertices of the two generating squares which are shown by hatching.

2. A screw and the corresponding nut, having a R. H. single-cut rectangular thread, are shown in Fig. 450. Since the cross section  $1-2-4-3$  of the protruding thread of the screw is the same as the cross section of the recess  $1-2-4-3$ , the sum of the dimension  $3_2-1_2$  of the protrusion and the dimension  $1_2-3_2$  of the recess is equal to the pitch  $h$  of the screw.

The protrusions and recesses consist of cylindrical surfaces, exterior and interior, generated by the movement of the sides  $1-3$  and  $2-4$  of the generating rectangles, and of two right helical

surfaces, or right helicoids, obtained by the movement of the sides 1-2 and 3-4 of the rectangles.

The construction of the screw and nut is carried out by the drawing helices with a pitch  $h$  through points 1, 2, 3 and 4. For this purpose two coaxial cylinders are constructed, the radii of the bases of which differ by the depth 1-2 of the thread recess. Having divided the circle of the base of the cylinders into a number of equal parts, say six, we divide the pitch  $h$  into the same number of equal parts and construct four helical lines, i. e., two on the inside cylinder and two on the outside. Having determined the visible outline of the protrusions and recesses of the thread we draw them in with continuous lines. The hidden outlines are usually shown only within the limits of one or two screw pitches  $h$ .

The cross-section of the screw cut by the horizontal plane  $\alpha$  is also shown in Fig. 450. This plane cuts the exterior and interior cylinders and two right helical surfaces. The section of the cylindrical surfaces are arcs of circles, the sections of the right helical surfaces are the diametrically opposite segments  $KL$  and  $MN$ . The horizontal projections  $K_1, L_1, M_1$  and  $N_1$  of points  $K, L, M$  and  $N$  are determined by drawing projectors from points  $K_2, L_2, M_2$  and  $N_2$ , in which the frontal trace of the cutting plane cuts the frontal projections of the helices. These projectors intersect the horizontal projections of the lateral cylinder surfaces in the points required.

The nut is represented in cross-section cut by a frontal plane  $\beta$  passing through the axis of the screw and nut. On this cross-section the visible part of the thread is seen as rising from right to left. This corresponds to the way in which the rear, hidden part of the R. H. thread of the screw rises, i. e., from right to left. The hidden lines shown in the cross-section of the nut are those parts of the rear helices which are covered by other parts of the thread.

3. A right double-thread, screw with a thread of triangular section is shown in Fig. 451. The screw is cut by a horizontal plane perpendicular to its axis.

The circles of the bases of the cylinders and the screw pitch  $h$  are divided into 12 equal parts. Then, the six helices, corresponding to the number of vertices of two generating triangles, are constructed.

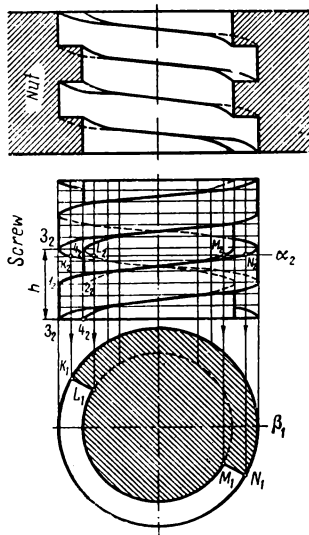


Fig. 450

The surface of this screw consists of two oblique helicoids. The frontal projections of the visible outlines on these surfaces are obtained by drawing tangents to the frontal projections of the exterior and interior helicoid lines. In Fig. 452 these outlines are shown separately on an enlarged scale.

To construct the section of the screw cut by plane  $\alpha$  auxiliary planes  $\beta$  and  $\gamma$  are drawn through the axis of the screw. These planes intersect each protrusion in triangles, the projections of which are shown in Fig. 451. The points  $K$  and  $L$ , in which plane  $\alpha$  intersects sides  $MN$  and  $M'N'$  of these triangles, are points contained in the lines of intersection of the screw and plane  $\alpha$ . Having constructed the horizontal projections

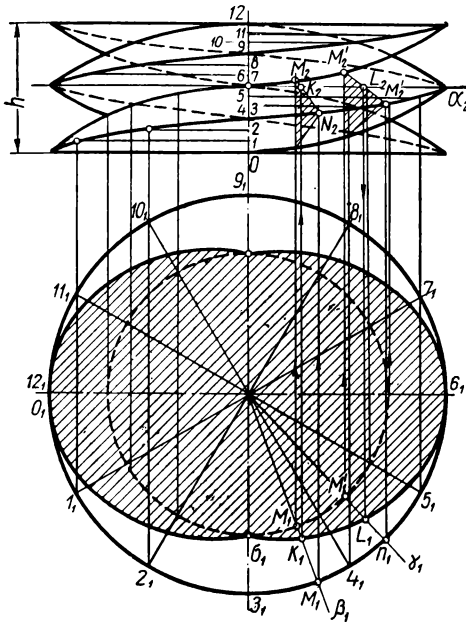


Fig. 451

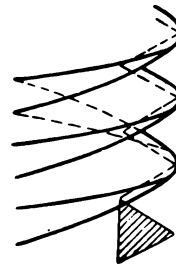


Fig. 452

$K_1$  and  $L_1$  of these points and joining the points  $6_1, K_1, L_1$  and  $6_1$  by a continuous line, we obtain the horizontal projection of one of the branches of the line of intersection of the screw and plane  $\alpha$ . The other three branches of this curve, which are Archimedean spirals, are constructed in a similar manner. The line of intersection has two axes of symmetry. The L. H. and R. H. parts of intersection are symmetrical because plane  $\alpha$  cuts a double-thread screw.

4. A single-cut R. H. screw, the cross section of the thread of which is an equilateral trapezoid, is represented in Fig. 453. The surface of such a screw is limited by two cylinders and two oblique helicoids. The cross-section of the screw given by cutting plane  $\alpha$  is also constructed with the help of auxiliary planes passing through the axis of the screw. The points  $K, L, Q$  and  $S$  belonging

to the outline of the section are given by the intersection of plane  $\alpha$  with the sides  $MN$ ,  $M'N'$ ,  $M''N''$ ,  $M'''N'''$  of the trapezoid along which the screw protrusions are intersected by the auxiliary axial planes. Since the screw in this case is single-cut, the intersection obtained is limited by arcs of two circles and two branches of the Archimedean spiral.

5. A screw and nut, with a thread of trapezoid section, are shown in Fig. 454. There are two generating trapezoids. The direction of the turns is R.H. The surface of the screw consists of two cylinders, right helicoids and oblique helicoids. The cross-section of the screw and nut given by the cutting plane  $\alpha$  is constructed with the help of auxiliary planes  $\beta$ ,  $\beta'$ ,  $\beta''$  and  $\beta'''$  which are passed through the axis of the screw. The circular parts of the outline of the section correspond to the parts of the right helicoid cut by plane  $\alpha$ . The curves, which are Archimedean spirals and lie between the outer and inner circles, are the sections of the oblique helicoids cut by plane  $\alpha$ .

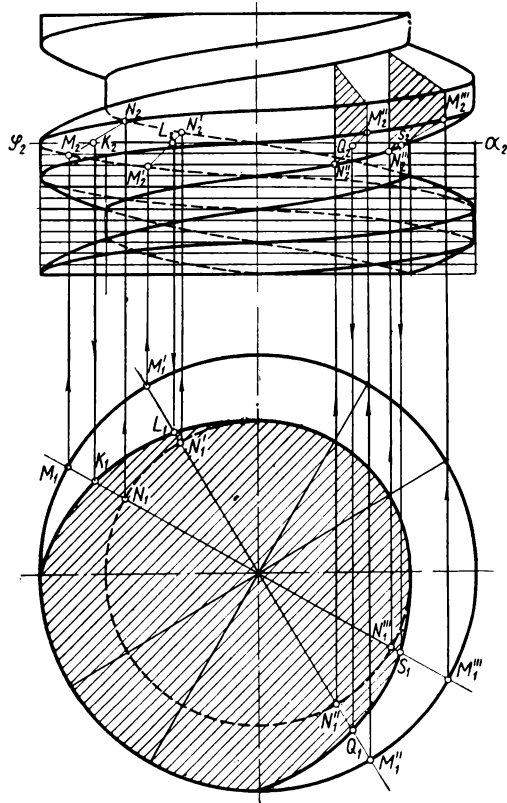


Fig. 453

6. A screw and nut with a R. H. triple-turn trapezoid thread is shown in Fig. 455. The cross-sections of the screw and nut given by plane  $\alpha$  consist of three equal parts symmetrically located.

In the section of the nut, seen on the left of Fig. 455, the exterior helix, which is covered by the helical surface in front of it, is shown as a broken line.



## § 52. Planes Tangent to Curved Surfaces

1. A plane may be tangent to a curved surface in one or more points, depending on the form of the surface. In the first case the tangent plane may be determined by drawing through the point of tangency, taken on the curved surface, two straight lines respectively tangent to two curved lines passing through this point and lying on the curved surface, for example, to two circles on the surfaces of a sphere (Fig. 456, *a*). An example of the second case is a plane which is tangent to a single-curved surface, such as a cylindrical surface, on which all the points of contact lie on one line, i. e., on a generatrix of the cylinder. This is the *line of tangency* (Fig. 456, *b*). The line of tangency is one of the lines determining the tangent plane. A second line of this plane is the tangent to any curved line on the given surface, such as the circular base of the cylinder.

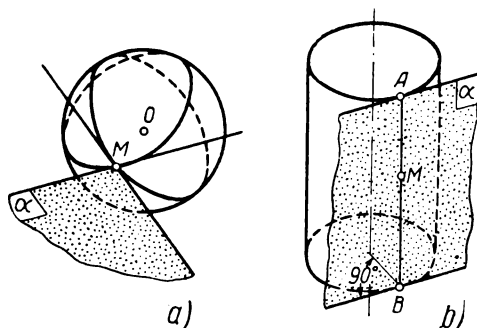


Fig. 456

2. Let it be required to draw a plane tangent to a cylindrical surface, passing through a given point  $M$  on the surface (Fig. 457).

The generatrix  $AB$  passes through point  $M$ . Through point  $B$ , i. e., the horizontal trace of this generatrix, draw line  $CD$  tangent to the circle of the base of the cylinder. The plane determined by the intersecting lines  $AB$  and  $CD$  is the required tangent plane.

The line  $CD$  is the horizontal trace of the tangent plane. To construct the frontal trace, which is not shown in the drawing, it is necessary to find the frontal traces or piercing points of two lines

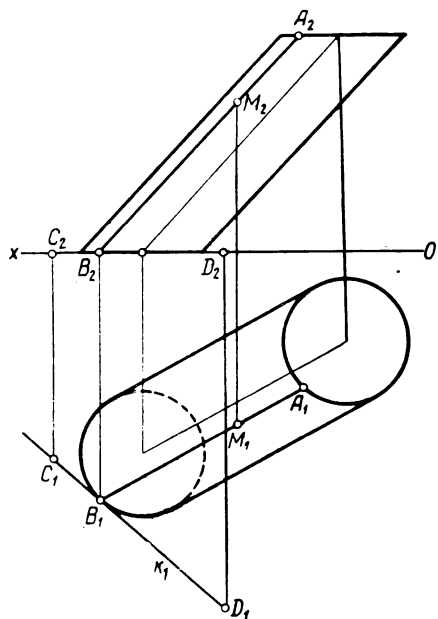


Fig. 457



contained in the plane, for example, the frontal trace of the generatrix  $AB$  and of a horizontal passing through point  $M$ . The frontal trace of the plane will pass through these frontal piercing points.

The plane tangent to the surface of a cone and passing through point  $M$  on the surface is constructed in a similar way (Fig. 458).

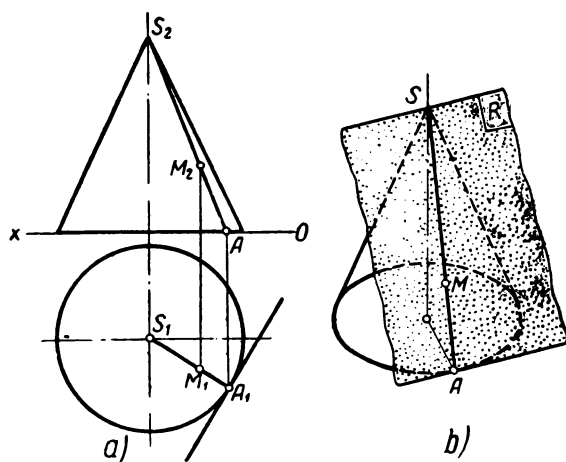


Fig. 458

3. A plane tangent to a spherical surface is perpendicular to the radius of the sphere at the point of tangency. Therefore, to construct a plane tangent to the sphere at a given point  $M$  on its surface (Fig. 459), it is sufficient to draw through point  $M$  a horizontal  $AB$  and frontal  $ED$  perpendicular to radius  $CM$  of the sphere. The plane defined by these intersecting lines will be the required tangent plane.

4. Let it be required to draw a tangent plane to a surface of revolution and passing through a given point  $M$  on the surface (Fig. 460). The method consists in drawing two curves on the surface of revolution through point  $M$ , the tangents to these curves at point  $M$  determining the required plane.

Through point  $M$  we draw a parallel, i. e., a horizontal section, and the tangent  $AB$  to this section at point  $M$ . Then a meridian is drawn. The frontal projection of this meridian is not shown in the drawing (Fig. 460). At point  $M$  a tangent to the curve should be drawn. In order to draw this tangent we can consider that the meridian, together with point  $M$ , is turned about the axis of the surface of revolution until it takes up a position parallel to plane  $\Pi_2$ . Point  $M$  will then occupy the position  $M'$ . The meridian will coincide with the principal meridian, i. e., with the generatrix to the right of the surface. Next the tangent  $CM'$  to the curve at

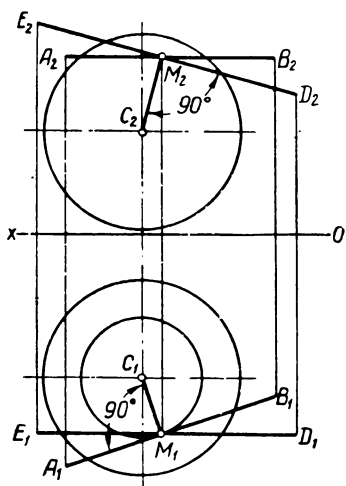


Fig. 459

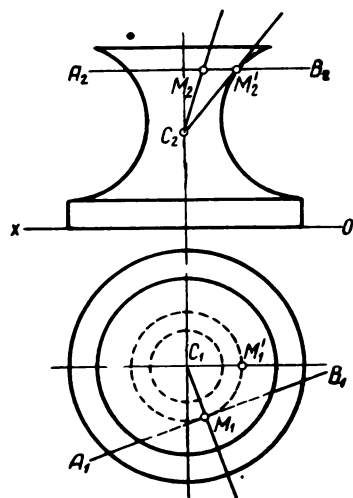


Fig. 460

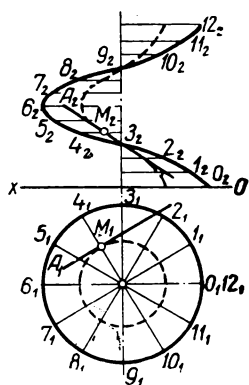


Fig. 461

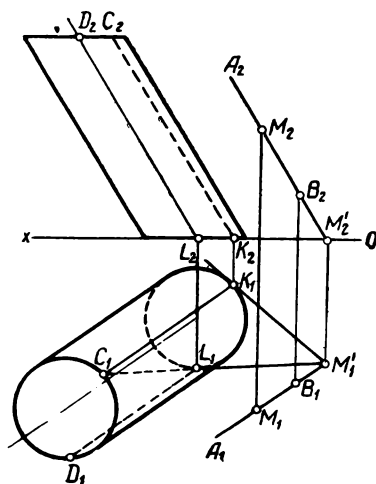


Fig. 462

point  $M'$  is drawn. With the reverse revolution of line  $CM'$  point  $C$  remains stationary, since it lies on the axis of revolution. The intersecting lines  $AB$  and  $CM$  define the required plane.

5. The construction of a plane tangent to a helical conoid in a given point  $M$  is seen in Fig. 461.

The generatrix  $M4$  is one of the two straight lines determining the required plane. As the second line  $AM$  we take the tangent to the coaxial helix, passing through point  $M$ .

6. Let it be required to construct a plane tangent to the surface of a cylinder, passing through a point  $M$  lying outside the surface (Fig. 462). The required plane is determined by the line  $AB$  drawn through point  $M$  parallel to the generatrices of the cylinder and by the tangent  $KM$  to the horizontal trace of the surface of the cylinder drawn through the horizontal piercing point  $M'$  of line  $AB$ . The line of tangency on the surface of the cylinder will be generatrix  $KC$ . Since a second tangent  $LM'$  may be drawn to the circle of the base from point  $M'$  this problem has two solutions. The tangents  $KM'$  and  $LM'$  are the horizontal traces of planes tangent to the surface of the given cylinder.

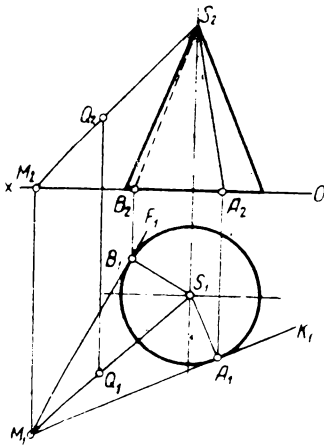
7. The construction of two planes tangent to the surface of a cone and passing through a point  $Q$  not contained in this surface is shown in Fig. 463. By joining point  $Q$  with the vertex  $S$  of the cone we find the horizontal piercing point  $M$  of line  $SQ$ . Through this point  $M$  we draw two tangents to the circle of the base of the cone. Lines  $M_1K_1$  and  $M_1F_1$  are the horizontal traces of the planes which pass through point  $Q$  and are tangent to the surface of the cone. Each of these traces together with line  $SM$  determines one of the tangent planes.

8. Let it be required to pass through a point  $Q$  a plane tangent to the surface of a sphere (Fig. 464).

This problem is solved by the construction of a conical surface with its vertex in point  $Q$  and described about the given sphere. A straight line is passed through the centre of the sphere  $C$  and point  $Q$ . Next we find the true length  $C_4Q_4$  of this line  $CQ$  on a new plane of projection  $\Pi_4$ . Having constructed the projection of the sphere on plane  $\Pi_4$ , through point  $Q_4$  we draw tangents to this projection. These tangents are the projections on plane  $\Pi_4$  of the limiting elements of a cone described about the sphere. By substituting plane  $\Pi_1$  by plane  $\Pi_5$  perpendicular to the axis of the cone, we obtain a circle with centre  $Q_5$ . This is the projection of the cone on the new plane  $\Pi_5$ .

Any plane tangent to the surface of this cone will be tangent to the given sphere and will pass through point  $Q$ . One of these

The plane of the triangle  $QNK$  can be constructed first in the system of planes of projection  $\Pi_4-\Pi_1$ , and then in the system of planes of projection  $\Pi_1-\Pi_2$ .



9. The construction of a tangent plane to the surface of a cylinder and parallel to a given line  $AB$  is shown in Fig. 465.

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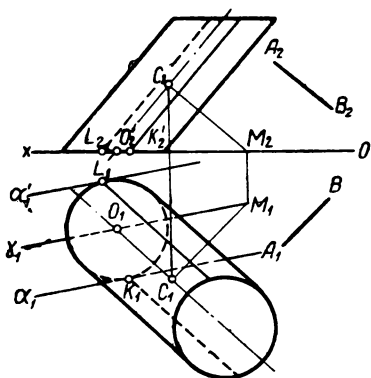


Fig. 465

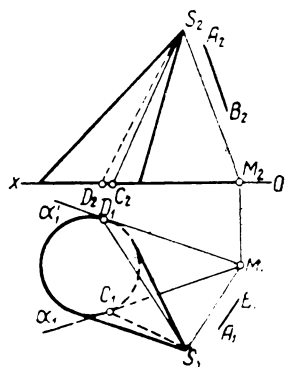


Fig. 466

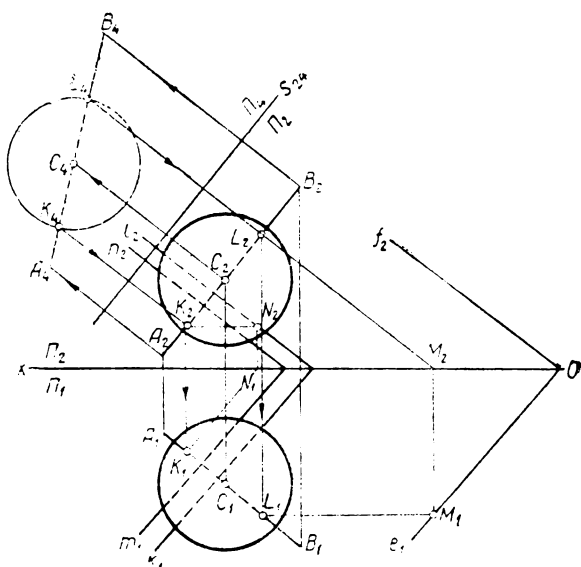


Fig. 467

10. Let it be required to draw a plane tangent to the surface of a cone and parallel to line  $AB$  (Fig. 466).

Through the vertex  $S$  of the cone we draw a line parallel to the given line  $AB$  and find the horizontal piercing point  $M$  of line  $SM$ . Then through point  $M_1$  we draw tangents to the circle of the base of the cone. These tangents will be the horizontal traces of planes tangent to the surface of the cone. These planes are parallel to line  $AB$  and the generatrices  $SC$  and  $CD$  are their lines of tangency on the surface of the cone. This problem has two solutions.

11. The construction of a plane tangent to the surface of a sphere and parallel to a given plane  $\gamma$  is shown in Fig. 467. Since a plane tangent to the surface of the sphere and parallel to the given plane  $\gamma$  must pass through the limiting point of a radius of the sphere drawn perpendicular to plane  $\gamma$ , this problem has two solutions.

We draw line  $AB$  through the centre  $C$  of the sphere perpendicular to plane  $\gamma$ . We then determine points  $K$  and  $L$  in which this line  $AB$  pierces the surface of the sphere. When line  $AB$  is projected onto a new plane of projection  $\Pi_4$ , to which it is parallel, it is seen in its true size. This new projection  $A_4B_4$  will intersect the projection of the surface of the sphere on plane  $\Pi_4$  in points  $K_4$  and  $L_4$ . These are the projections of the extreme points of diameter  $KL$ , which is perpendicular to plane  $\gamma$ . Having constructed the projections of points  $K$  and  $L$  on planes  $\Pi_2$  and  $\Pi_1$ , as shown by arrows, we draw the required planes  $\alpha$  and  $\alpha'$  through points  $K$  and  $L$  parallel to the given plane  $\gamma$ . This is done with the help of the horizontal  $KN$  and frontal  $LM$ , also parallel to plane  $\gamma$ .

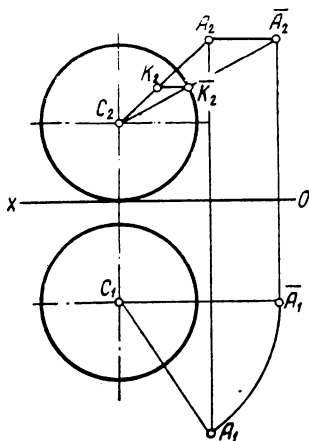


Fig. 468

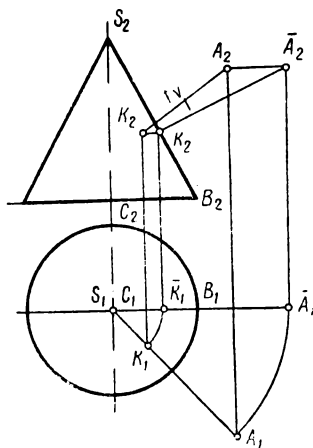


Fig. 469

**12. Normals to curved surfaces.** In general, when constructing a normal to a curved surface at a point on the surface, using the procedure described above, it is first necessary to draw a plane tangent to the surface through the given point. After that a perpendicular to this plane is drawn through the point.

In certain cases normals to curved surfaces are drawn without auxiliary lines, for example, the normal to the surface of a sphere in a given point passes through this point and the centre of the sphere.

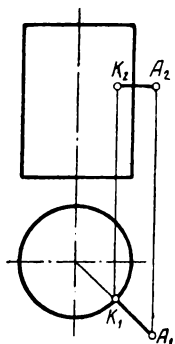


Fig. 470

The solution of the problem of determining the distance from point  $A$  to the surface of sphere is seen in Fig. 468. This distance is given by the length of the normal  $AK$ . Point  $K$  is the piercing point of line  $AC$  on the surface of the sphere. The construction is carried out using the method of revolution.  $\overline{A_2K_2}$  is the required distance.

The shortest distance from a given point  $A$  to the surface of a cone (Fig. 469) is also determined by drawing a normal to the conical surface.

Point  $A$  is rotated about the axis  $SC$  of the cone to the position  $\overline{A}$ . The radius of revolution  $CA$  then occupies the frontal position  $\overline{CA}$ . From point  $\overline{A_2}$  we drop a perpendicular to the frontal projection  $S_2B_2$  of the generatrix  $SB$ . Point  $\overline{K}$ , the intersection of the perpendicular and generatrix  $SB$ , is the point on the surface of the cone nearest to point  $\overline{A}$ , so  $\overline{A_2K_2}$  is the required distance. Lines  $A_1K_1$  and  $A_2K_2$  are the projections of the normal  $AK$ .

In Fig. 470 is shown the determination of the shortest distance from point  $A$  to surface of cylinder. This distance  $AB$  projects onto plane  $P_1$  in its true length.

#### Problems and Exercises

1. How many planes can be passed through a given point tangent to a surface of a) a cylinder, b) a cone and c) a sphere?
2. Is it possible, in general, to pass a plane tangent to the surface of a) two cones, b) a cylinder and a cone?
3. Draw a tangent plane through a given point on the surface of an elliptical cone.
4. Draw a tangent plane to the surface of a torus passing through a given point on its surface.
5. Draw a tangent plane to an oblique helicoid passing through a given point on its surface.
6. Through point  $M$  taken outside an oblique cone draw a tangent plane to its surface.
7. Draw a plane  $\alpha$  tangent to the surface of a right circular cone parallel to a line  $AB$ .

## APPENDIX

### The Use of Models in Descriptive Geometry

The special difficulty, which the beginner as a rule experiences, is to be able to conceive clearly the connection between the figures drawn on paper and the actual figures in space which they represent.

In other words, when reading the text where the constructions on a plane drawing are explained, it is necessary to imagine as taking place in space the various operations used to solve the problem on paper with the help of points, lines and plane figures.

This difficulty may be largely avoided by making use of models at the beginning of the course.

The drawings enabling models to be made for 24 constructions are given in the book. The problems are arranged so that by using the multiview drawings and the axonometric representation the student, without help of the teacher, can study the development of surfaces and the way to lay out patterns for them.

It has been found that by using models very clear and precise concepts are formed from the start and a correct method of working is acquired.

The drawings given include models of planes of projection, projecting planes and geometrical figures with lines and projections plotted on them.

The dotted lines indicate the edges to be cut and the allowances to be left along the various sides of the pattern for overlap and where to glue the projecting planes to the planes of projection. Each pair of planes of projection should be fixed at right angles with the help of narrow strips of paper as shown in the drawing for making Fig. 10. Sometimes, for example in the case of the model suggested for Fig. 80, the planes of projection may be fixed at right angles without using strips.

To represent horizontal-projecting lines match sticks may be used. If a match is not long enough two may be joined. A convenient



way of doing this is to make a small cut in the end of one match while the other is sharpened into the form of a spatula. This is then dipped in glue and inserted into the cleft of the first match.

When matches are supposed to stand vertically the end to be glued to the horizontal plane of projection should be cut clean and square. The matches should be held in place until the glue hardens sufficiently to hold them in a vertical position.

Bases are not always necessary. Sometimes, as for example in the model for Figs. 57 and 59, the projection line may be fixed with the help of holes made in the planes of projection which are slightly lengthened.

The student may himself make other models for those constructions which seem most difficult. Nevertheless it should not be forgotten that excessive use of models may hinder him from visualizing projections on the drawings.

The student should endeavour to understand the drawings and be able to solve any problems involving projections, no matter how complex, by applying the methods of descriptive geometry without having often to resort to models.

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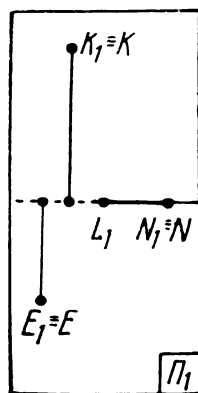
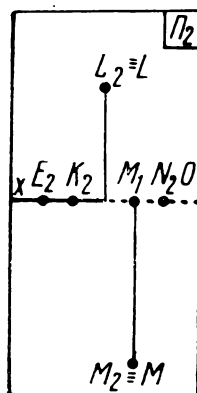
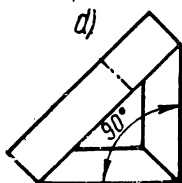
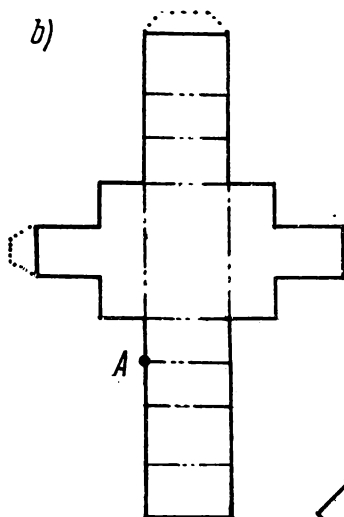
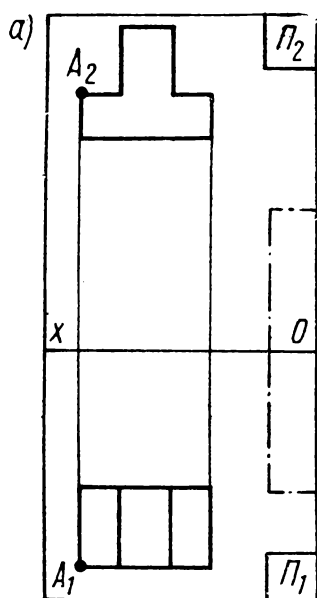


Fig. 471 to Fig. 10

Fig. 472 to Fig. 20

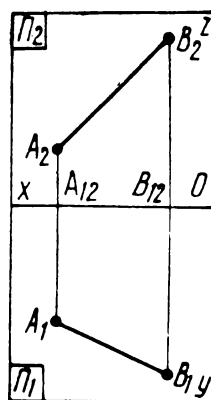
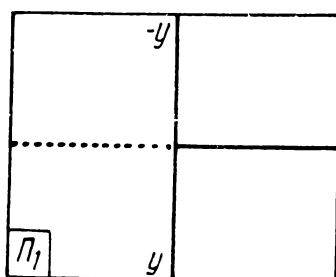
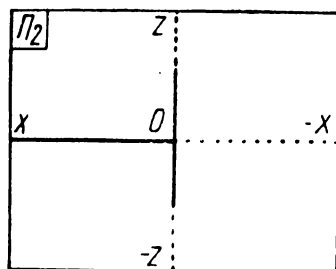


Fig. 474 to Fig. 36

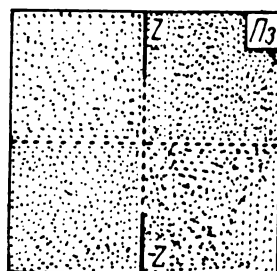


Fig. 473 to Fig. 21

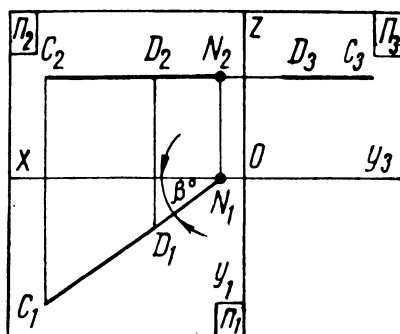


Fig. 475 to Fig. 39

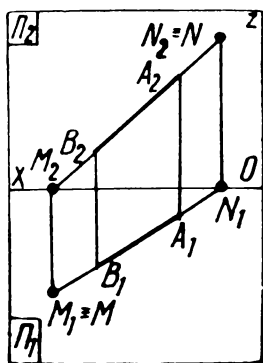


Fig. 476 to Fig. 57

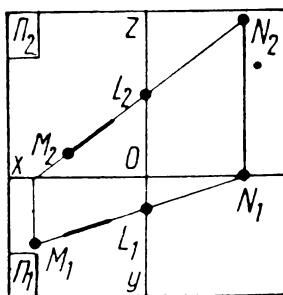


Fig. 477 to Fig. 59

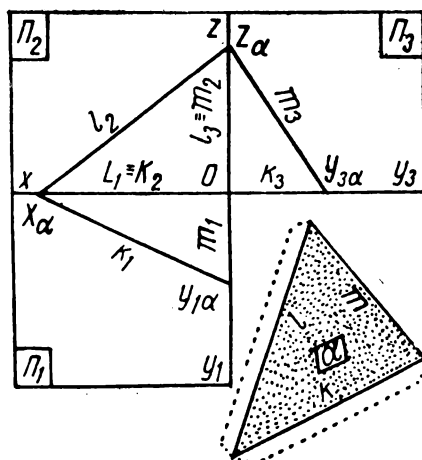
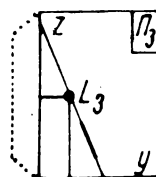


Fig. 478, to Fig. 80

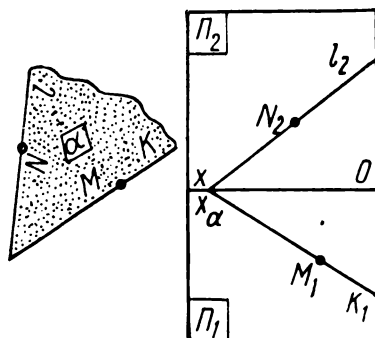


Fig. 479 to Fig. 86

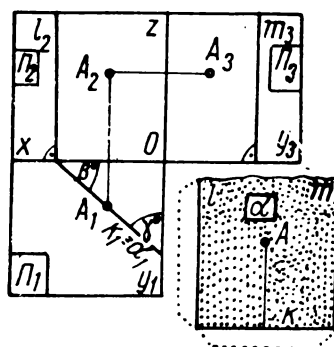


Fig. 480 to Fig. 90

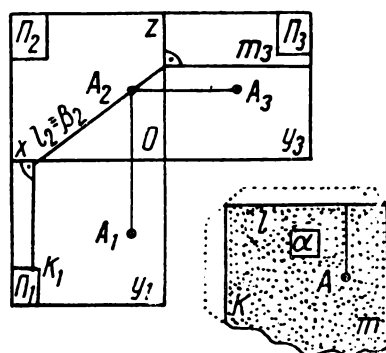


Fig. 481 to Fig. 92

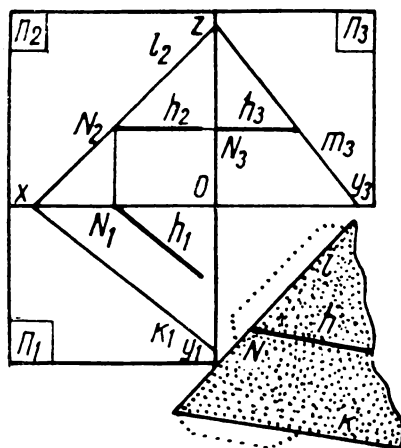


Fig. 482 to Fig. 104

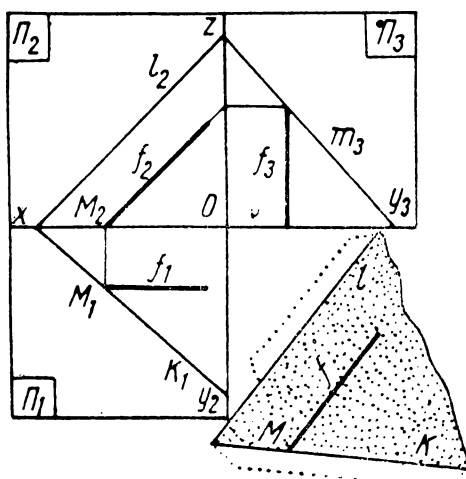


Fig. 483 to Fig. 105

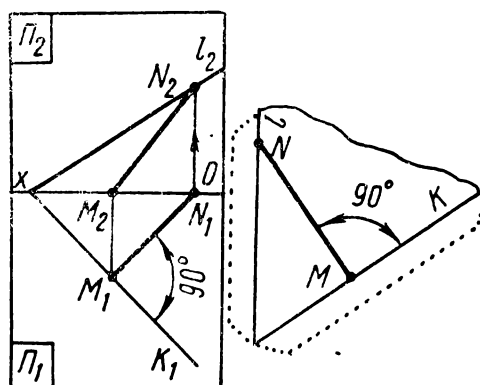


Fig. 484 to Fig. 116

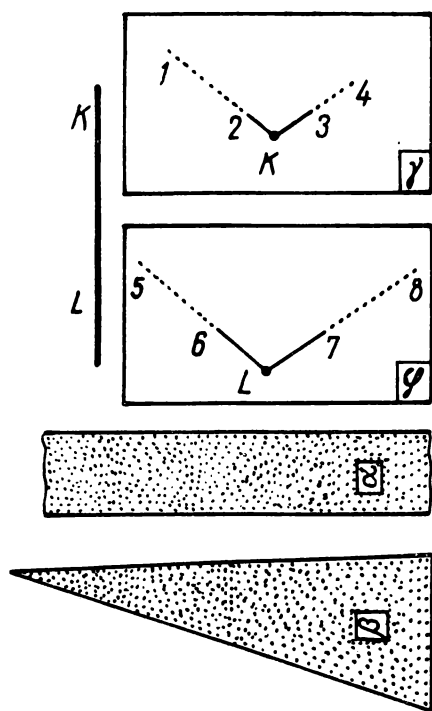


Fig. 485 to Fig. 149

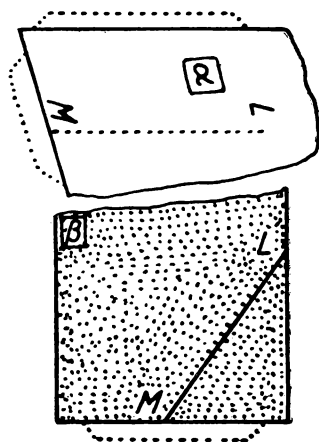
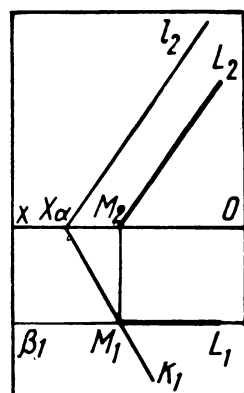


Fig. 486 to Fig. 154

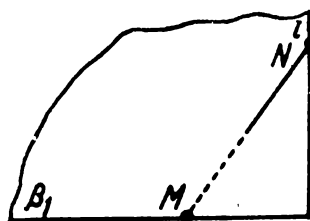
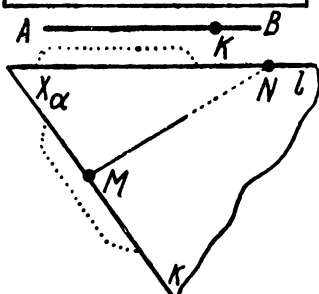
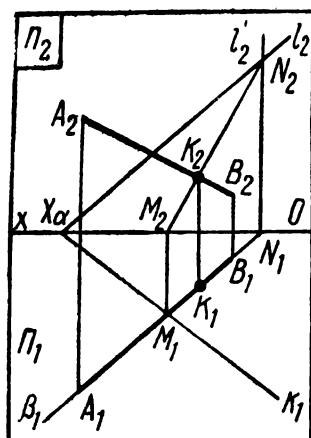


Fig. 487 to Fig. 166

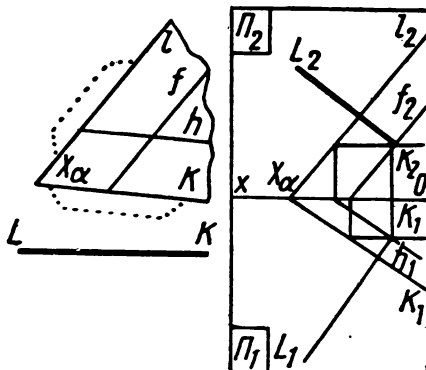


Fig. 488 to Fig. 177

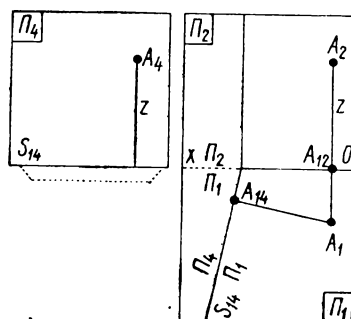


Fig. 489 to Fig. 198



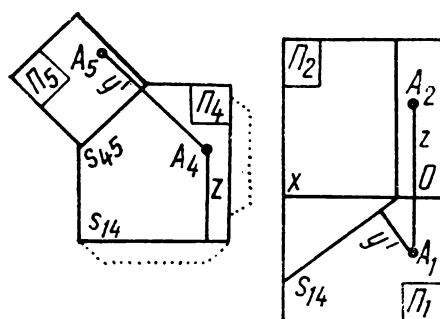


Fig. 490 to Fig. 208

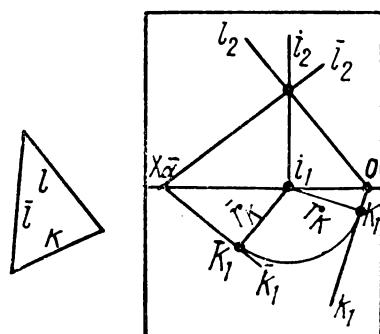


Fig. 491 to Fig. 247

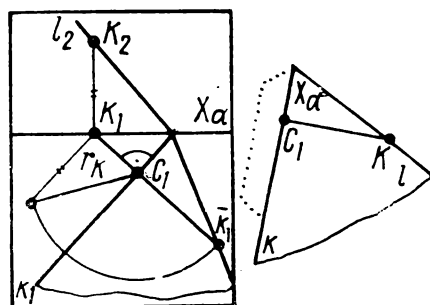


Fig. 492 to Fig. 269

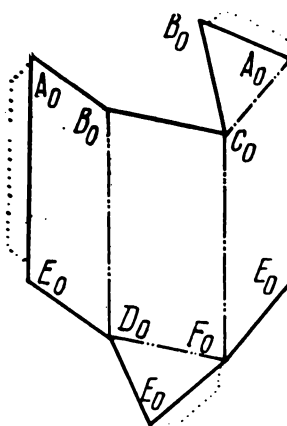
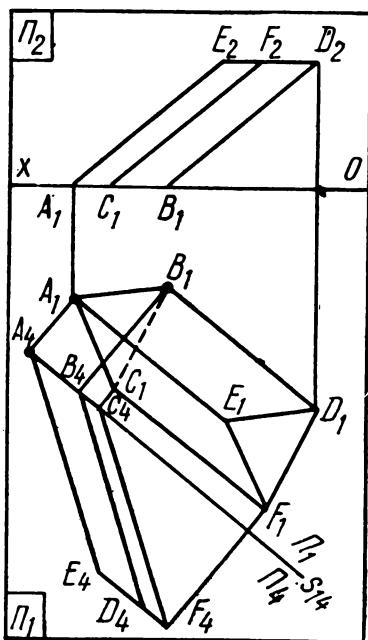


Fig. 493 to Fig. 292

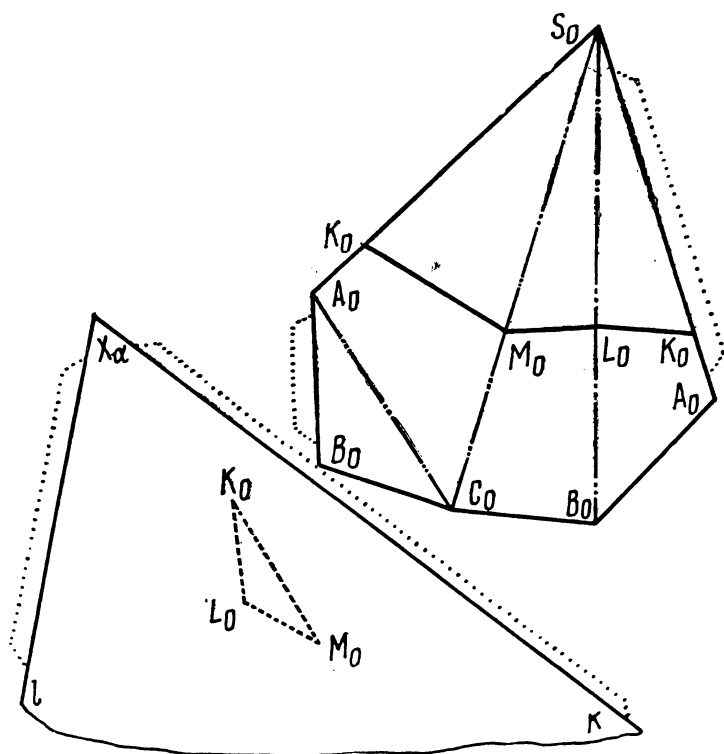


Fig. 494 to Fig. 311

А. Т. ЧАЛЫЙ

КУРС НАЧЕРТАТЕЛЬНОЙ ГЕОМЕТРИИ  
(на английском языке)







